Discrete Mathematics

Sets

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Contents

Set

A **set** is an unordered collection of its **elements** (or members).

The set is fully specified by its elements.

Usually capital letters are used to name sets and lowercase letters for naming variables representing their elements.

Denotations:

- $x \in X$: "x is an element (or member) of the set X", "element x belongs to the set X", etc.
- $x \notin X$: "x is not an element of the set X"

Sets, cont.

A set that contains finitely many elements is called “finite”, otherwise it is called an “infinite” set.

The order of elements in the set **does not matter**:
e.g. $\{2,3,5\} = \{3,5,2\}$

Each element of a set has multiplicity of 1 (1 copy) [1]

[1] Multiple copies are possible in so-called *multi-sets*

Written forms

Two written forms for defining the set:

- listing all the elements of the set **inside braces separated by commas**:
  e.g. X = {0,1,2,3,4,5,6,7,8,9}
- defining the set by its members that have some *properties*.
  form: { universe : properties }
  e.g. Y = {natural numbers x : x is even and x is greater than 20}
  (read as: "natural numbers x **such that** x is even and x is greater than 20")

The set in the last example is **infinite** (i.e. it contains infinitely many elements)

Examples:
$2 \in X$, $10 \notin X$, $10 \notin Y$, $100 \in Y$

Empty set

Empty set: $\emptyset$ is the set that contains **no elements**

There exists only 1 empty set.

Universal set

Almost anything can be an element of a set (a set does not have any pre-specified "type".
In particular, sets can be elements of other sets (but not of itself!).

However, to avoid some *logical paradoxes* it is often assumed that all the potentially considered elements of the sets are restricted to some pre-specified domain called **universal set** (or universe).

Universal set is usually mentioned in the "set builder" notation, in the first part of the definition, e.g.:

{natural number x : x is even and x is greater than 20}
here: the universe is the set of all natural numbers

Convention:

The word "family" is often used to emphasise that some set has sets as its members. (e.g. "a family of all subsets of a given set", etc.)

Inclusion

One set X is **contained** (or included) in another set Y if and only if all the elements of X are also the elements of Y.

$X \subseteq Y \Leftrightarrow x \in X \rightarrow x \in Y$

We say "X is a subset of Y" and "Y is a superset of X" (or "Y contains X")

If a subset X of Y is not equal to Y we can call it a **proper subset** of Y and can use slightly different denotation: $X \subset Y$.

In particular, any set is contained in itself: $X \subseteq X$
(because the statement $x \in X \rightarrow x \in X$ is always true)

Empty set is contained in any set (also in itself): $\emptyset \subseteq X$
(exercise: why? (use the definition and properties of the implication)

Any set is contained in the universe: $X \subseteq U$.

Equality of sets

Two sets are equal (identical) if and only if **they have the same elements**:

$X = Y \Leftrightarrow (x \in X \leftrightarrow x \in Y)$

Note: Two sets X, Y are equal iff $X \subseteq Y$ and $Y \subseteq X$ (this can serve as a convenient general scheme to prove set equality!)

Union of sets

The union of two sets X and Y, denoted as $X \cup Y$ is the set so that each element of it is the member of X or the member of Y (or both):

$X \cup Y = \{x : x \in X \vee x \in Y\}$

Intersection of sets

The intersection of two sets X and Y, denoted as $X \cap Y$ is the set so that each element of it is the member of both X and Y:

$$X \cap Y = \{x : x \in X \wedge x \in Y\}$$

Symmetric difference of sets

The symmetric difference of sets X and Y, denoted as $X \oplus Y$ is the set so that each element of it is either a member of X or a member of Y, but not both.

$X \oplus Y = \{x : x \in X \oplus x \in Y\}$ (exclusive or, “xor”)

Complement of set

A complement of a set X is the set of all members of the universe U that are not members of X:

$x \in X' \Leftrightarrow (x \in U \wedge \neg(x \in X))$

The empty set and the universe are mutually complements of themselves.

$\emptyset' = U, \; U' = \emptyset$

Complement of complement of a set X is X (and any even number of complements as well):
$(X')' = X$

a small exercise:
prove it from the definitions of complement, set equality, negation, etc.

Difference of two sets

The difference of sets X and Y, denoted as $X \setminus Y$ is the set so that the each element of it is a member of X and is not a member of Y:

$$X \setminus Y = \{x : x \in X \land x \notin Y\}$$

notice that: $X' = U \setminus X$

Set identities

For any sets A,B,C and the universal set U (containing them), the following identities hold:

Identity:	Name:
$A \cup \emptyset = A,\ A \cap \emptyset = \emptyset$	identity laws
$A \cup U = U,\ A \cap \emptyset = \emptyset$	domination laws
$A \cup A = A,\ A \cap A = A$	idempotent laws
$A'' = A$	complementation law
$A \cup B = B \cup A,\ A \cap B = B \cap A$	commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	distributive laws
$(A \cup B)' = A' \cap B',\ (A \cap B)' = A' \cup B'$	De Morgan's laws
$A \cup (A \cap B) = A,\ A \cap (A \cup B) = A$	absorption laws
$A \cup A' = U,\ A \cap A' = \emptyset$	complement laws

Examples of predefined number sets

There are conventional denotations for some **number sets**:

- $\mathcal{B} = \{0, 1\}$ (bit values)
- $\mathcal{N} = \{0, 1, 2, 3, ...\}^*$ (natural numbers)
- $\mathcal{Z} = \{..., -2, -1, 0, 2, 1, ...\}^*$ (integer numbers)
- $\mathcal{Q} = \{x : \exists_p \exists_q\, p \in \mathcal{Z} \land q \in \mathcal{Z} \land x = p/q \land q \neq 0\}$ (rational numbers)
- $\mathcal{R}$ (real numbers)[2]
- $\mathcal{IQ} = R \setminus Q$ (irrational numbers)

Notice that $\mathcal{B} \subset \mathcal{N} \subset \mathcal{Z} \subset \mathcal{R} \supset \mathcal{IQ}$

* the notation with "..." is a bit informal as it is not precise.

[2] The formal definition of real numbers is a bit technically involved and we skip it here

Ordered pair

An ordered pair of two elements (x, y) is a 2-element sequence (x is called the first element of the ordered pair and y is called the second element).

Difference to a two-element set $\{x, y\}$:

- the order matters: $(x, y) \neq (y, x)$ (unless $x = y$)
- x can be equal to y and it is still a pair, i.e. (x, x)

Cartesian product of sets

The **Cartesian product** of two sets X and Y denoted as $X \times Y$ is the set of all possible ordered pairs so that the first element is a member of X and the second element is the member of Y:

$$X \times Y = \{(x, y) : x \in X \land y \in Y\}$$

Power set

The set of all subsets of a given set X (including the empty set) is called the **power set** of X and denoted as 2^A (or $P(A)$).

Example:
$2^{\mathcal{B}} = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$

Remark: if all the members of some set are other sets, we can use the words *family of sets*. Thus, we can say: "$P(A)$ is the family of all subsets of X".

Venn diagrams

Some operations and properties can be illustrated on pictures with so-called *Venn diagrams*.

Each set on a Venn diagram is denoted with a circle on a picture and represented by all the points that are inside the circle.

Summary

- Sets: basic definitions and notation
- Set operations
- Set identities
- pair and Cartesian product of sets
- power set
- Venn diagrams

Example tasks/questions/problems

- for given 2 small finite sets compute all the presented operations on these sets
- observe analogies between logical operators and operators and relations from set theory (list at least 5 of them!)
- observe analogies between propositional logic identities and set identities (list at least 5 such analogies)
- using the definitions of set operations and logical tools (such as propositinal identities, etc.) prove selected 5 of the presented set identities
- compute the power set of the set $X = a, b, c, d$ (how many elements does it have?)

Thank you for your attention.