Discrete Mathematics

Predicates

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Natural number 2 is even.

Natural number 3 is even.

Predicate

Let U be some universal set.

A logical expression containing some *variable* that becomes a proposition when we *substitute* any particular value from the universe for this variable is called a **predicate**.

It is also called *propositional function*.

It is usually denoted similarly to functions, for example:

$\phi(x)$, $f(x)$, $\psi(x)$, etc.

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What does it mean in short?

x is divisible by 6

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There are more values for a that make the above propositions true.

Predicate Logic

Predicate logic is more powerful than propositional logic as it can express more.

Predicate can be viewed as an extension of proposition by introducing *variables*. Predicates, similarly to propositions, can be atomic or compound, for example:

Predicate as an expression of “property”

Predicate can be also viewed as a description of some *feature* of the object from the universe represented by the variable, e.g. having some color, being positive, etc.

Examples:

“$x > 0$”, (“being a positive number”) (`positive(x)`)

“x is green” (“having a green color”) (`green(x)`)

The above predicates are not propositions, since the truth value depends on the value of the variables that are not (yet) specified.

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"$yx > 0$ AND x is integer AND y is integer" ("x and y are integers of the same sign")

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`isCapitalOf(Warsaw, Poland)`

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`playedBridge(a,b,c,d)`: *people a,b,c,d played bridge together*

Free variables of a predicate

Unspecified variables of a predicate are called its *free variables*.

Examples:
P(x): "x > 0" (x is a free variable)
"$yx > 0$ AND x is integer AND y is integer" (x and y are free)

Predicate can become a proposition when there are no free variables.

This may happen in two basic ways:

- *substituting* some concrete value for a variable
- *quantifying* a variable with a **quantifier**

Quantifiers

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Note: Each quantifier has its natural **range** (or scope) of the variable it bounds, i.e. the part of the predicate associated with it. If a variable of the same name is outside of its range it is still free. E.g.:

$(\forall_x P(x)) \lor Q(x)$
(x is still a free variable in this compound predicate!)

Universal quantifier: examples

$P(x)$: x is a student, universe: all people in this room

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P(x): "x > 0"

$\forall_x P(x)$: "for each x, x is positive"

Examples

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NOTICE: the choice of domain is necessary to determine the truth value of the (now) proposition

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Examples:
domain: integers
$\exists_x P(x)$
"there exists integer that is positive" (true)

domain: real numbers less than -1
$\exists_x P(x)$ "there exists an integer less than -1 that is positive" (false)

Logical equivalence of predicates

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Example:
$\forall_x P(x) \land \forall_x Q(x) \equiv \forall_x (Q(x) \land P(x))$

the above is an example of a logical equivalence (general law) since it holds for any domain of the variable x.

Notice: the quantifiers have higher precedence than any other logical operators.

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$\exists x\forall y\exists z\neg P(x,y,z)$

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- quantifying x: $\forall_x G(x, y)$ is a predicate with 1 free variable ("each natural number is not less than y")

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Now, for example:

- substitution of 0 for y: G(x,0) is a predicate with 1 free variable ("x is nonnegative")
- quantifying x: $\forall_x G(x, y)$ is a predicate with 1 free variable ("each natural number is not less than y")
- mix of substition and quantification, e.g. $\forall_x G(x, 0)$ is now a **proposition** (no free variables) "any natural is non-negative" (true!), etc.

Examples

$$\forall\epsilon\,\exists\delta\,\forall h\,(|h| < \delta) \Rightarrow (|f(x+h) - f(x)| < \epsilon)$$

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(continuity of function f)

Examples

$$\forall\epsilon\,\exists m\,\forall n\,|a_{m+n} - b| < \epsilon$$

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(convergence of a sequence a to limit b)

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Examples

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(convergence of a sequence a to limit b)

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(convergence of a sequence of functions f_i to a function f)

Examples

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Example: "big O" asymptotic notation

$$f(n) = O(g(n))$$

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$$\exists c\, \exists n_o\, \forall n\, (n > n_0) \Rightarrow f(n) \leq c \cdot g(n)$$

Generalised unions and intersections

Imagine we have a family $\mathcal{F}$ of sets indexed by a set of indices I (that can be infinite!):

$\mathcal{F} = \{F_i : i \in I\}$

The **generalised union** of F, denoted as $\bigcup_{i \in I} F_i$ is defined as follows:

$\bigcup_{i \in I} F_i = \{x : \exists_{i \in I}\, x \in F_i\}$

The **generalised intersection** of F, denoted as $\bigcap_{i \in I} F_i$ is defined as follows:

$\bigcap_{i \in I} F_i = \{x : \forall_{i \in I}\, x \in F_i\}$

Summary

- Predicates
- Quantifiers
- Negation
- Logical equivalence of predicates
- Nested quantifiers
- Generalised set operations

Example tasks/questions/problems

- definition of predicate, free variable
- definition and interpretation of quantifiers (2 kinds)
- negating the quantifiers (including nested ones)
- practice translating mathematical concepts to the predicate logic. E.g.:
    - using only logical and mathematical symbols "=", "*" define a predicate Q(x,y) that reads as "x is a divisor of y" for the domain of integer variables x,y (hint: you can use helper variables and quantifiers)
    - using only logical symbols and "<" and "=" write a predicate specifying that x (a variable) is a maximum in some set S of numbers (being the domain)
- give examples of predicates P(x), Q(x) and domain that show that the two following statements: $(\forall_x P(x) \lor Q(x))$ and $\forall_x P(x) \lor \forall_x Q(x)$ are not logically equivalent. Does any of them **implies** the other? (i.e. is any of them a **stronger** condition?)

Thank you for your attention.