Discrete Mathematics

Functions

(c) Marcin Sydow

Contents

Definition of a function

Definition: A relation $f \subseteq X \times Y$ is called a **function** if and only if for each element of $x \in X$ there exists *exactly one* $y \in Y$ so that $(x, y) \in f$.

Function is denoted as follows:

$$f : X \to Y$$

X is called the **domain** of f and Y is called **co-domain** of f.

Functions, cont.

Elements of X (domain) are called *arguments* and elements of Y (co-domain) are called *values* of the function.

Since there is exactly one value ofor each argument, it is possible to write:

$$f(x) = y$$

for particular $x \in X$ and $y \in Y$.

Function is also called *mapping* of X into Y.

Set of all functions

Set of all possible functions that have domain X and co-domain Y is denoted as:

$$Y^X$$

Defition of function written with mathematical symbols

Definition (in natural language): A relation $f \subseteq X \times Y$ is called a **function** if and only if for each element of $x \in X$ there exists *exactly one* $y \in Y$ so that $(x, y) \in f$.

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$$\forall_{x \in X} [\exists_{y \in Y} (x, y) \in f]$$

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The resulting expression:

$$\forall_{x \in X}[\exists_{y \in Y}(x, y) \in f] \wedge [\forall_{y,y' \in Y}((x, y) \in f \wedge (x, y') \in f) \Rightarrow y = y']$$

Example

$X = \{x \in N : x < 5\}$
Is the following a function?

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Equality of functions

Two functions $f : X \to Y$ and $g : A \to B$ are **equal** iff the following conditions hold:

- $X = A$ (equality of domains), $Y = B$ (equality of co-domains)
- $\forall_{x \in X} f(x) = g(x)$

Restriction and extension of a function

Let $f : X \rightarrow Y$ and $f' : X' \rightarrow Y$ and $X \subseteq X'$

If f(x)=f'(x) for all $x \in X$ we say that **f' is an extension of f** and **f is a restriction of f'**

Graph of a function

Given a function $f : X \to Y$ if the set of pairs $f = \{(x, y) \in X \times Y : y = f(x)\}$ can be naturally mapped to points in the plane with Cartesian coordinates (e.g. when X=Y=R), we can view the f as its **graph**.

Injection

A function $f : X \rightarrow Y$ is **injection** iff the following holds:

$$\forall_{x,x' \in X}\, x \neq x' \Rightarrow f(x) \neq f(x')$$

An injection is also called a “one-to-one” function.

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Is the following an injection?

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$f : Z \to Z, \ f(x) = x^2$

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$f : Z \rightarrow Z,\ f(x) = x^2$

$f : N \rightarrow N,\ f(x) = x^2$

Surjection

A function $f : X \rightarrow Y$ is **surjection** iff the following holds:

$$\forall_{y \in Y} \exists_{x \in X}\, y = f(x)$$

A surjection is also called "onto mapping" (or "f maps X *onto* Y")

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$f : R \rightarrow (0, 1), \ f(x) = 1/(1 + e^{-x})$

Bijection

A function $f : X \rightarrow Y$ is **bijection** iff it is injection and surjection.

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Is the following a bijection?

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Inverse of a function

If $f : X \rightarrow Y$ is an injection, then the **inverse** of this function is the (unique) function $f^{-1} : Y \rightarrow X$ defined as follows:

$$f^{-1}(y) = x \Leftrightarrow f(x) = y$$

questions:

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is inverse of injection an injection? (yes)
is inverse of bijecion a bijection? (yes)

Example of inverse

$f : R \rightarrow (0,1)$, $f(x) = 1/(1+e^{-x})$
$f(x) = x^2$ for non-negative reals
$f(x) = 2^x$, for non-negative reals

Composition of two functions

For two functions $f : X \to Y$ and $g : Y \to Z$ their **composition** is the function $g \circ f : X \to Z$ defined as follows for any $x \in X$:

$$(g \circ f)(x) = g(f(x))$$

(Notice the order of the functions in the denotation $g \circ f$)

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Notice: non-commutativitiy and associativity of compositon

Definition of (infinite) sequence

A **sequence** $a_0, a_1, a_2, a_3, \ldots$ is a function whose domain is the set of natural numbers $\mathcal{N}$ $a : \mathcal{N} \rightarrow X$, where X is some set.
For any number $i \in \mathcal{N}$ $a(i)$ is usually denoted as a_i.
In particular, if X is a number set, the sequence is *numeric* (e.g. for $X = \mathcal{R}$ it is a real sequence.

Image of a set

For a function $f : X \rightarrow Y$ and a set $A \subseteq X$ the **image** of A is the set $f(A) \subseteq Y$ defined as follows:

$$f(A) = \{y \in Y : \exists_{x \in A} \, y = f(x)\}$$

(to avoid misunderstanding of the denotation $f(A)$ we assume that $A \notin X$)

Inverse image of a set

For a function $f : X \to Y$ and a set $B \subseteq Y$ the **inverse image** of B is the set $f^{-1}(B) \subseteq X$ defined as follows:

$$f^{-1}(B) = \{x \in X : f(x) \in B\}$$

(to avoid misunderstanding of the denotation $f^{-1}(B)$ we assume that $B \notin Y$)

Image of union

Assume that $f : X \to Y$.
For any sets $A, A' \subseteq X$ the following holds:

$$f(A \cup A') = f(A) \cup f(A')$$

Image of intersection

Assume that $f : X \rightarrow Y$.
For any sets $A, A' \subseteq X$ the following holds:

$$f(A \cap A') \subseteq f(A) \cap f(A')$$

(the equality does not hold in general: example?)

Example

$f : Z \to N$, $f(x) = x^2$

A is the set of negative integers, A' is the set of positive integers.

What is $A \cap A'$?

What is $f(A \cap A')$?

What is $f(A)$?

What is $f(A')$?

What is $f(A) \cap f(A')$?

Image of intersection cont.

What property of the function f would suffice for the equality:

$$f(A \cap A') = f(A) \cap f(A')$$

?

Image of intersection cont.

What property of the function f would suffice for the equality:

$$f(A \cap A') = f(A) \cap f(A')$$

?

The above equality holds if the function f is an injection.

Image of difference

Assume that $f : X \to Y$.
For any sets $A, A' \subseteq X$ the following holds:

$$f(A \setminus A') \subseteq f(A) \setminus f(A')$$

(the equality does not hold in general: example?)

Inverse image of union

Assume that $f : X \to Y$.
For any two sets $B, B' \subseteq Y$ the following holds:

$$f^{-1}(B \cup B') = f^{-1}(B) \cup f^{-1}(B')$$

(notice: we do not assume that f is an injection)

Inverse image of intersection

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Composition of image and inverse image

Assume that $f : X \to Y$.
For any $A \subseteq X$ the following holds:

$$A \subseteq f^{-1}(f(A))$$

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For what conditions the equality holds?

Composition of image and inverse image

Assume that $f : X \to Y$.
For any $A \subseteq X$ the following holds:

$$A \subseteq f^{-1}(f(A))$$

For what conditions the equality holds? (for f being an injection)
i.e. if f is an injection then $A = f^{-1}(f(A))$.

Composition of image and inverse image, cont.

For any $B \subseteq f(X)$ the following holds:

$$f(f^{-1}(B)) = B$$

Composition of image and inverse image, cont.

For any $B \subseteq f(X)$ the following holds:

$$f(f^{-1}(B)) = B$$

Why the assumption $B \subseteq f(X)$ above is important? (give an example)

Thank you for your attention.