Discrete Mathematics

Equipollence

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Introduction

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the set $X = \{a, b, c, d\}$ has 4 elements. How many elements does the following set have? $Y = \{1, \emptyset, \{1, 2\}, \{\{\emptyset\}, 2\}\}$

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Equipollence Relation between two sets

Two sets X and Y are **equipollent** if and only if there exists a *bijection* $f : X \to Y$ between them (i.e. a function that is injection and surjection).

Denotation: $X \sim Y$

Example: $X = \{a, b, c, d\}$
$Y = \{\{1\}, \emptyset, \{1, 2\}, \{\{\emptyset\}, 2\}\}$

Are X and Y equipollent? Is there also a bijection $g : Y \to X$ in this case?

Reminder: any bijection $f : A \to B$ has its *inverse* $f^{-1} : B \to A$, defined as $f^{-1}(b) = a \Leftrightarrow f(a) = b$

Equipollence Relation is Equivalence Relation

Equipollence relation between 2 sets is equivalence relation, since it satisfies:

- Reflexivity: for any set X it holds that: $X \sim X$ (is equipollent with itself) (why?)

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Hence, equipollence relation is equivalence relation (thus, it has equivallence classes)

Cardinality of a set

Cardinality (or **cardinal number**) of a set is defined as its **equivalence class** in terms of equipollence relation, i.e. two sets have the same cardinality (or cardinal number) if they are equipollent.

Example:
The sets $X = \{a, b, c, d\}$, $Y = \{\{1\}, \emptyset, \{1, 2\}, \{\{\emptyset\}, 2\}\}$ have the same cardinality. It is also the same as for the set $A = \{1, 2, 3, 4\}$

Note: For finite sets cardinality means the same as the number of elements of the set.

But now we can also talk about cardinality of **infinite** sets!

Countable sets

A set X is **countable** if and only if it equipollent with the set of natural numbers $\mathcal{N}$ (infinitely countable) or its finite subset (finitely countable).

The cardinal number of infinite countable sets is denoted as $\mathfrak{a}$ (or equivalently as $\aleph_0$ (**aleph zero**))

Examples

Is the following set countable (finitely/infinitely)?

What is its cardinal number?

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- odd natural numbers

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- all natural numbers greater than 100

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- odd natural numbers yes/$\aleph_0$
- all natural numbers greater than 100 yes/$\aleph_0$

Examples of countable sets

Are the following sets countable? (i.e. are they equipollent with the set of natural numbers)?:

- the set of all integers?
- the set of all possible ordered pairs of natural numbers?
- the set of all pairs of integers?
- the set of rational numbers?

Interpretation of Countability

Countability (or cardinality of $\aleph_0$) can be informally viewed as **the smallest infinity number**.

In this sense, the set of natural numbers is the smallest infinite set.

Important: a countable set X has the following (equivalent) properties:

- all the elements of X can be arranged in a sequence (why?)
- it is possible to "process" all the elements of X one after another in some sequential order so that each separate element will be processed in *finite* time

Important: there exist sets that are not countable.
("larger" sets than $\mathcal{N}$)

Countability vs set operations

The following statements are true:

- the union $A \cup B$ of any two countable sets A and B is countable (why?)
- the Cartesian product $A \times B$ of any two countable sets A and B is countable (why?)

The above statements can be (by mathematical induction) generalised to any finite number of countable sets.

The set of all possible finite sequences of terms belonging to a countable set is countable

Uncountable sets

Any infinite set which is not equipollent with $\mathcal{N}$ is called **uncountable**.

Interpretation: uncountable set is a set that has "larger" cardinality than $\mathcal{N}$. All the elements of uncountable set cannot be arranged in a sequence!

Real numbers are uncountable

Theorem:

The set of all real numbers is uncountable.

Proof: It suffices to show that for every sequence $a_1, a_2, ...$ of real numbers, there exists a real number x that does not belong to this sequence.

Theorem: *The real interval* $(0, 1)$ *is uncountable*

Proof: example of a "diagonal proof".

Proof cont.

Lets define a sequence of closed intervals $[p_i, q_i]$ so that:

- $q_i - p_i = 1/3^i$
- $[p_i, q_i] \subseteq [p_{i-1}, q_{i-1}]$
- $a_i \notin [p_i, q_i]$

For example, in the closed interval $[0, 1]$ let's define $[p_1, q_q]$ as a one of the three: $[0, 1/3], [1/3, 2/3], [2/3, 1]$ that does not contain a_1. Next, inside the interval $[p_1, q_1]$ let's define subinterval of length $1/9$ that does not contain a_2, etc.

Now, let x be the intersection $\bigcup_{i=1}^{\infty}[p_i, q_i]$ (which is non-empty, since the intervals are closed). Hence, $\forall_i x \neq a_i$ because $a_i \notin [p_i, q_i]$ while $x \in [p_i, q_i]$.

Continuum

The cardinality of the set of real numbers is denoted as:

$$\mathfrak{c} = |R|$$

and called **continuum**.

Definition of Addition and multiplication

For any disjoint sets X, Y we define the operations on their cardinalities as:

$$|X| + |Y| = |X \cup Y|$$

For any sets X, Y:

$$|X| \cdot |Y| = |X \times Y|$$

Note: the sets can be infinite or even uncountable

Examples

The following properties of aleph zero hold:

- $\mathfrak{a} + \mathfrak{a} = \mathfrak{a}$
- $\mathfrak{a} \cdot \mathfrak{a} = \mathfrak{a}$
- $\mathfrak{a} + n = \mathfrak{a}$
- $\mathfrak{a} \cdot n = \mathfrak{a}$

(n is any finite natural number)

Proof: exercise

Properties of cardinal numbers

For any three cardinal numbers, the associative and distributive law for addition and multiplication hold:

- $(\mathfrak{m} + \mathfrak{n}) + \mathfrak{p} = \mathfrak{m} + (\mathfrak{n} + \mathfrak{p})$
- $(\mathfrak{m} \cdot \mathfrak{n}) \cdot \mathfrak{p} = \mathfrak{m} \cdot (\mathfrak{n} \cdot \mathfrak{p})$
- $\mathfrak{m}(\mathfrak{n} + \mathfrak{p}) = \mathfrak{m} \cdot \mathfrak{n} + \mathfrak{m} \cdot \mathfrak{p}$

Exponentiation of cardinal numbers

For any two cardinal numbers $\mathfrak{m} = |X|$, $\mathfrak{n} = |Y|$ the exponentiation $\mathfrak{n}^{\mathfrak{m}}$ is defined as the cardinality of the set of all functions $f : X \to Y$ (reminder: denotation of this set is Y^X).

$$\mathfrak{n}^{\mathfrak{m}} = |Y^X|$$

Properties of exponetiation

The following formulas hold for any three cardinal numbers:

- $\mathfrak{n}^{\mathfrak{m}+\mathfrak{p}} = \mathfrak{n}^{\mathfrak{m}} \cdot \mathfrak{n}^{\mathfrak{p}}$
- $(\mathfrak{m}\mathfrak{n})^{\mathfrak{p}} = \mathfrak{m}^{\mathfrak{p}} \cdot \mathfrak{n}^{\mathfrak{p}}$
- $(\mathfrak{n}^{\mathfrak{m}})^{\mathfrak{p}} = \mathfrak{n}^{\mathfrak{m}\mathfrak{p}}$

Cardinality of the power set

Theorem:

For any set X of cardinality $\mathfrak{m}$ the cardinality of its power set (the family of all its subsets) denoted as 2^X is equal to the cardinal number $2^{\mathfrak{m}}$.

Proof: consider the *characteristic function* $f : X \rightarrow \{0, 1\}$ of the subset $S \subseteq X$ (1 if the element belongs to S, 0 otherwise).

Cantor Theorem

Theorem:

No set has cardinality equal to that of the family of all its subsets. Equivalently: $2^{\mathfrak{m}} \neq \mathfrak{m}$ *for any cardinal number* $\mathfrak{m}$

Set of all sets does not exist

Theorem:

The set of all sets does not exist.

Proof 1: it is a corollary from the Cantor theorem, since the cardinality of family of all subsets of any set X is different than the cardinality of any subsets of X. Family of all subsets of the "set of all set" (if it existed) would be a subset of it.

Note: it means that the "collection" of all sets is not a set.

Russel antinomy

(another proof that the "set of all sets" cannot exist)

Let's assume the following set is possible to be defined:

$$Z = \{x : x \notin x\}$$

Now: does Z belong to itself or not?

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Now: does Z belong to itself or not?

($x \in Z \Leftrightarrow x \notin x$ what is equivalent to $Z \in Z \Leftrightarrow Z \notin Z$ - contradiction!)

Note: this can be viewed as "warning" on the limits of the concept of the set.

Inequality of cardinal numbers

We define: for $\mathfrak{m} = |X|$, $\mathfrak{n} = |Y|$

$$\mathfrak{m} \leq \mathfrak{n} \Leftrightarrow X \subseteq Y$$

If additionally $\mathfrak{m} \neq \mathfrak{n}$ then we denote it as $\mathfrak{m} < \mathfrak{n}$.

Note: Cantor theorem is equivalent to:

$$\mathfrak{m} < 2^{\mathfrak{m}}$$

Cantor-Bernstein Theorem

Theorem:

$$\mathfrak{m} \leq \mathfrak{n} \wedge \mathfrak{n} \leq \mathfrak{m}$$

implies that:

$$\mathfrak{m} = \mathfrak{n}$$

Note: it is equivalent to say that the relation is antisymmetric. It implies that this relation on cardinal numbers defines a linear order

Example tasks/questions/problems

For each of the following: precise definition or ability to compute on the given example (if applicable):

- equipollence relation
- cardinality
- what is aleph zero
- what is continuum
- operations on cardinal numbers
- draft of proof of uncountability of real numbers (or interval (0,1))
- Cantor theorem
- Cantor-Bernstein theorem

Thank you for your attention.