Discrete Mathematics

Counting

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Contents

Introduction

The topic of this lecture is counting (or enumerating) different configurations of some discrete objects with some specified constraints, such as sets, sequences, functions, etc.

Counting has important applications in computer science and mathematics, for example:

- determining complexity of algorithms
- computing discrete probability of some events

Examples of applications of counting problems

Counting can be applied to computing (for example):

- how many passwords of a given lenght and specified constraints on the used symbols are potentially possible
- how many different IP addresses are possible in a given protocol (e.g. IPv4 or IPv6)
- how many different telephone numbers in a given country or network are possible
- how many different diving licence plates in a given country are possible
- the probability of any event concerning counting, such as any of the above examples and similar
- the number of operations to be executed by a loop (or a nested loop) in a computer program
- how many different chemical molecules with a given properties are possible
- how many different ways of finding a path in a given graph are possible, etc.

Abstract representation of the counting problems

Most of the concrete counting problems can be represented as some frequently encountered general abstract counting schemes that include:

- product rule and sum rule
- pigeonhole principle (Dirichlet drawer principle)
- inclusion-exclusion principle
- counting permutations and combinations (with or without repetitions)
- generating functions[1]

Mastering the general abstract schemes as the above makes it possible to practically solve variety of concrete counting/enumerating problems.

[1] not in this lecture

General Counting Method

Solving most of the practical counting problems consists of 2 steps:

1. recognise what general abstract pattern (or a combination of patterns) represents the practical problem under study to be solved
2. apply appropriate technique to solve the problem represented by the discovered pattern or a combination of patterns

The representation method

Another way of viewing the general method of counting is as follows.

C is the set of the objects to be enumerated.

Find an abstract mathematical representation of the counted objects so that

- there is a bijection between the set of enumerated objects and the set of representations
- it is easy to compute (or known) what is the number of different representations

The representations could be, for example: sets, sequences, functions, injections, permutations, etc.

(examples)

Product Rule

If a procedure of doing something can be split into $k = 2$ separate sub-procedures that are done *mutually independently* so that:

- the first sub-procedure can be done in exactly n_1 ways
- the second sub-procedure can be done in exactly n_2 ways

Then the whole procedure can be done in exactly $n_1 \cdot n_2$ ways

Generalisation: the product rule can be generalised to any $k > 2$. In such case, the number of ways is equal to $\prod_{i=1}^{k} n_i = n_1 \cdot n_2 \cdot \ldots \cdot n_k$

Denotation for the number of elements of a set

For any finite set X its number of elements (called "power of the set" or "cardinality of the set") is denoted as:

$$|X|$$

Example: $|\{a,b,c,d\}| = 4$, $|\emptyset| = 0$, etc.

Product Rule in terms of set theory

The product rule is equivalent to the following simple fact:

If A_1, A_2 are 2 finite sets, then the number of elements of their Cartesian product[2] is given by the following formula:

$$|A_1 \times A_2| = |A_1| \cdot |A_2|$$

i.e. it is the product of the number of elements of A_1 and A_2.

(example)

In the product rule for counting the set A_1 represents the set of ways of doing the sub-procedure 1 and A_2 the set of ways of doing the subprocedure 2.

(example)

[2] Reminder: Cartesian product $A \times B$ is the set of all possible ordered pairs (a, b) such that $a \in A$ and $b \in B$.

Generalisation of the product rule

Mathematical induction can be used to prove the following implication of the product rule for sets:

$$|A_1 \times \ldots \times A_n| = \Pi_{i=1}^{n}|A_i| = |A_1| \cdot \ldots \cdot |A_n|$$

where for all $0 \geq i \geq n$ A_i is some finite set.

(example)

Sum Rule

If a procedure of doing something can be done **either** in a one of n_1 ways **or** in a one of n_2 ways, then the procedure can be done in $n_1 + n_2$ ways.

(example)

Sum rule in terms of set theory

If A_1, A_2 are two **disjoint** finite sets then the following holds:

$$|A_1 \cup A_2| = |A_1| + |A_2|$$

(example)

In the context of the sum rule for counting the sets A_1, A_2 represent the sets of mutually excluding ways of doing the procedure.

The Inclusion-Exclusion Principle for 2 sets

For any two finite sets the following holds:

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

(example)

General Inclusion-Exclusion Principle

For any family of $n > 1$ finite sets the following holds:

$$|\bigcup_{i=1}^{n} A_i| = \sum_{k=1}^{n}(-1)^{k-1} \sum_{1 \leq i_1 < \ldots < i_k \leq n} |A_{i_1} \cap \ldots \cap A_{i_k}|$$

examples:

- (n = 2): the same as the Inclusion-Exclusion Principle for two sets.
- (n = 3): $|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_2 \cap A_3| - |A_1 \cap A_3| + |A_1 \cap A_2 \cap A_3|$
- (n = 4): (exercise)

Example of application

What is the number of the positive integers not higher than 100 that are not divisible by any of the following numbers: 3,5,7?

Hint: consider the 3 sets of positive numbers not higher than 100 not divisible by 3,5,7 respectively and apply the general Inclusion-Exclusion Principle.

Pigeonhole Principle

If $k+1$ objects are placed into k boxes, for any $k \in N^{+}$, at least one box contains more than 1 object.

Proof: (by contraposition) Assume that all of the k boxes contain not more than 1 object. Then the total number of objects is not higher than k.

It is also called "Dirichlet Drawer Principle"

Example: Among eight people at least two of them have birthday on the same day of the week.

Example 2: There exist at least two persons that have the same number of hair on their heads.

Generalised Pigeonhole Principle

If n objects are placed into k boxes, then at least one box contains at least $\lceil N/k \rceil$ objects.

Proof: (by contraposition) similar as the previous one.

Example: among 1000 people at least 3 have exactly the same birhtday.

Permutations

Permutation of an n-element set is a n-element sequence (order matters) of its elements so that every element is used exactly once.

Example: permutations of the set $\{1, 2, 3\}$:
$(1, 2, 3), (1, 3, 2), (2, 1, 3), (2, 3, 1), (3, 1, 2), (3, 2, 1)$

Counting the Permutations

The number of permutations is given by the following expression:

$n! = \prod_i^n i = 1 \cdot 2 ... \cdot n$ (factorial)

Example: the number of ordering 10 digits is equal to $10! = 3628800$.

Proof (by the representation method and product rule): each permutation can be represented as a n-element sequence of distinct terms out of n-element set. The number of ways of constructing such a sequence can be viewed as follows: there are n ways of taking the first element of the sequence, n-1 ways (the remaining elements) for the second, n-2 for the third, etc.

Permutations with repetitions

The number of m-element sequences (order is important) where each of the terms has one of n possible values (called "m-permutations of an n-element set") is given by the expression:

n^m

Example: The number of possible 10-element strings of lower-case latin letters of alphabet is equal to 26^{10}

Counting the number of subsets of a given set

The number of subsets of a finite n-element set X is given by the expression:

$$2^n$$

(reminder: the family of all subsets of a set X is denoted as 2^X and called "power set of X")

$|2^X| = 2^{|X|}$ (example)

Proof

Proof (by representation rule and product rule): Let give numbers: 1,...,n to the elements of the set X. Each subset can be represented (representation rule) as a **characteristic vector** of this subset (i.e. binary vector $(b_1, ..., b_n)$ where each b_i is 1 (the i-th element is taken) or 0 (else). There is a bijective mapping between subsets and such characteristic vectors (each sequence represents exactly one subset and each set has exactly one characteristic vector. Hence, it is equivalent to count all the possible n-element characteristic vectors, but this is easy (product rule): the first element of the vector can be constructed in 2 ways (0 or 1), the second as well, etc. so that there are exactly $2 \cdot ... \cdot 2 = 2^n$ characteristic vectors.

Counting the number of functions

The set of all functions $f : X \rightarrow Y$ is denoted as Y^X The number of different functions $f : X \rightarrow Y$ is given by the expression $|Y^X| = |Y|^{|X|}$.

Proof: (by representation rule and product rule) each function can be represented as the $|X|$-element sequence where the i-th term is (one of $|Y|$ possible) value of the i-th argument. There is a bijective mapping between the functions and the sequences. But the number of the sequences is $|Y| \cdot ... \cdot |Y| = |Y|^{|X|}$ (by product rule).

Counting the number of injections

The number of different injections $f : X \to Y$ (where X, Y are some finite sets) is given by the expression:

$$y \cdot (y-1) \cdot (y-2) \cdot \ldots \cdot (y-x+1) = y!/(y-x)!$$

where $y = |Y|$ and $x = |X|$, assuming that $y \geq x$[3]

Example: how many injections $f : X \to Y$?:

- $X = \{a, b, c, d\}$, $Y = \{1, 2, 3, 4, 5\}$
- $X = \{a, b, c\}$, $Y = \{1, 2, 3\}$
- $X = \{a, b, c, d\}$, $Y = \{1, 2, 3\}$

Proof: (by representation rule and product rule)

[3] if $x > y$ there is no injection from X to Y

Partial permutations

The number of r-element sequences (without repetitions) of elements of an n-element set is given by the following expression (assume $r < n$):

$$P(n,r) = n!/(n-r)! = n \cdot (n-1) \cdot \ldots \cdot (n-r+1)$$

Proof: (by representation and product rule)

Notice that this is equivalent to injections from n-element set to r-element set.

Combinatorial Proof

A **combinatorial proof** is a general and powerful method for proving mathematical equations especially concerning the counting problems.

For any equation it is enough to find the way of explaining (a kind of short "story") that each side represents the same counting problem but differently stated.

Example: Binomial theorem.

In the left side of the expression the coefficient is equal to the number of ways of forming the k-th power of the variable x.

Binomial coefficient

The **binomial coefficient**, for two numbers $n, k \in N$, $k \leq n$ is defined as:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

It is read as "n choose k".

$$\binom{n}{k} = \binom{n}{n-k}$$

Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

(example)

Proof: combinatorial proof

k-element subsets of a given set (k-combinations)

The number of k-element subsets of an n-element set is given by the following expression:

$$C(n,k) = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Proof: (by representation and product rule) Name the elements of the set by numbers 1,...,n. Any k-element injective (without repetitions) sequence of such numbers represents some k-element subset (there are $n!/(n-k)!$ such sequences). But this representation is not bijective since exactly $k!$ different sequences represent the same subset. Hence, the number must be divided by $k!$.

Combinations with repetitions

The number of r-element combinations with repetitions out of a n-element set:

$$C(n+r-1,r) = C(n+r-1,n-1)$$

Proof: (by representation rule) Each such combination can be represented as a (n+r-1)-element sequence where the r elements are separated by n-1 separators.

Pascal's triangle

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Proof: combinatorial proof and sum rule

Notice: it is a recursive definition of the binomial coefficient

Sum of binomial coefficients

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Proof: combinatorial proof (hint: count subsets of an n-element set)

Another identity

$\sum_{i=0}^{n}(-1)^k\binom{n}{k} = 0$

Proof: substitute x=1 and y=-1 in the binomial theorem.

Another example:

$\sum_{i=0}^{n} 2^k\binom{n}{k} = 3^n$

Vandermonde's identity

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k} \binom{n}{k}$$

Proof: (combinatorial proof)

Multinomial Coefficient

For any $n \in N$ and $n_1, ..., n_k \in N$ so that $n = n_1 + n_2 + ... + n_k$ the **multinomial coefficient** is defined as:

$$\binom{n}{n_1, ..., n_k} = n!/(n_1! \cdot n_2! \cdot ... \cdot n_k!$$

Example: (permutations with indistinguishable objects, where there are n_i identical objects for each $0 \leq i \leq k$)

Example: balls in boxes

r identical balls in n distinguishable boxes:

$$C(n + r - 1, n - 1)$$

Summary

- Introduction
- Product and Sum Rule
- Representation Rule
- Pigeonhole Principle
- Inclusion-Exclusion Principle
- Combinatorial Proof
- Permutations and Combinations
- Permutations and Combinations with Repetitions
- Binomial Coefficitient

Example tasks/questions/problems

- Describe the concept of the: Product/Sum Rule, Pigeon Principle, Inclusion-Exclusion Principle, Representation Rule, Combinatorial Proof
- Give the definition of the permutation, combination, binomial coefficient
- Give the formula for counting: the number of k-element sequences out of n-element set (with/without repetitions), the number of all the subsets and of k-element subsets of a given n-element set, the number of functions/injections between two given finite sets, the number of combinations with repetitions
- provide and prove the discussed properties of the binomial coefficient

Thank you for your attention.