Calculus with Fuzzy Numbers

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Abstract. Algebra of ordered fuzzy numbers (OFN) is defined to handle with fuzzy inputs in a quantitative way, exactly in the same way as with real numbers. Additional two structures: algebraic and normed (topological) are introduced to define a general form of defuzzyfication operators. A useful implementation of a Fuzzy Calculator allows counting with the general type membership relations.

1 Introduction

Nowadays, fuzzy approach is helpful while dealing with non-exact data involving human vagueness in large multimedia databases. In real-life problems both parameters and data used in mathematical modelling are vague. The vagueness can be described by fuzzy numbers and sets.

Communication, data mining, pattern recognition, system modelling, diagnosis, image analysis, fault detection and others are fields where clustering plays an important role. However, in practice constructed clusters overlap, and some data vectors belong to several clusters with different degrees of membership. A natural way to describe this situation results in implementing the fuzzy set theory [18,21], where the membership of a vector $\mathbf{x}_k$ to the i-th cluster U_i is a value from the unit interval $[0, 1]$. Approximate reasoning, on the other hand, by means of compositional fuzzy rules of inference can help in dealing with uncertainty inherent in the processed knowledge, especially when building a fuzzy decision support system for decision making tasks in different domains of applications [25,27,28].

All those situations require well known fuzzy logic and even more – arithmetics of fuzzy numbers, in order to perform operations on fuzzy observations. Fuzzy concepts have been introduced in order to model such vague terms as observed values of some physical or economic terms, like pressure values or stock market rates, that can be inaccurate, can be noisy or can be difficult to measure with an appropriate precision

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because of technical reasons. In our daily life there are many cases where observations of objects in a population are fuzzy. In modern complex and large-scale systems it is difficult to adopt the systems using only exact data. Also in this case, it is inevitable to adopt non-exact data involving human vagueness. In this way we are approaching the concept of the fuzzy observation.

The commonly accepted theory of fuzzy numbers [1] is that set up in [4] by Dubois and Prade in 1978, who proposed a restricted class of membership functions, called (L, R)–numbers with two so-called shape functions: L and R. (L, R)–numbers can be used for the formalisation of basic vague terms. However, approximations of fuzzy functions and operations are needed if one wants to follow the Zadeh's extension principle [26,27]. It leads to some drawbacks that concern properties of fuzzy algebraic operations, as well as to unexpected and uncontrollable results of repeatedly applied operations [23,24].

Classical fuzzy numbers (sets) are convenient as far as a simple interpretation in the set-theoretical language is concerned. However, we could ask: How can we imagine fuzzy information, say X, in such a way that by adding it to the fuzzy number A the fuzzy number C will be obtained? In our previous papers (see [14] for references) we tried to answer that question in terms of so-called *improper parts* of fuzzy numbers. In this paper we consider the algebra of ordered fuzzy numbers that leads to an efficient tool in dealing with unprecise, fuzzy quantitative terms.

2 Motivations

One of the goals of our paper is to construct a revised concept of a fuzzy number, and at the same time to have the algebra of crisp (non-fuzzy) numbers inside the concept. The other goal is to preserve as much of the properties of the classical so-called *crisp reals* ***R*** as possible, in order to facilitate real world applications as e.g. in fuzzy control systems. The new concept allows for utilising the fuzzy arithmetic and constructing an algebra of fuzzy numbers. By doing this, the new model of fuzzy numbers has obtained an extra feature which was not present in the previous ones: neither in the classical Zadeh's model, nor in the more recent model of so-called convex fuzzy numbers. This feature, called in [12,14] the orientation, requires a new interpretation as well as a special care in dealing with ordered fuzzy numbers. To avoid confusion at this stage of development, let us stress that any fuzzy number, either classical (crisp or convex fuzzy) or ordered (new type), has its *opposite number* which is obtained from the given number by multiplication with minus one. For the new type of fuzzy numbers, multiplication by a negative real not only affects the support, but also the orientation swaps. It is important that to a given ordered fuzzy number two kinds of opposite elements are defined: the classical, one can say an algebraic opposite number (element) obtained by its multiplication with a negative crisp one, and the complementary number which differs from the opposite one by the orientation. Relating to an ordered fuzzy number, its opposite and complementary elements make the calculation more complex, however with new features. On the one hand a sum of an ordered fuzzy number and its algebraic opposite gives a crisp zero, like in the standard algebra of real number. On the other hand the complementary number can play the role of the opposite number in the sense of the Zadeh's model, since the sum of the both – the (ordered fuzzy) number and its complementary one – gives a fuzzy zero, non-crisp, in general.

We have to admit that the application of the new type of fuzzy numbers is restricted to such real-life situations where also the modelled circumstances provide information about orientation. In particular, in majority of existing approaches, for a fuzzy number A the difference $A - A$ gives a fuzzy zero. However, this leads to unbounded growth of the support of fuzziness if a sequence of arithmetic operations is performed between two (classical) fuzzy numbers. To overcome this unpleasant circumstance the concept of the orientation of a fuzzy number has been introduced as well as simple operations between those new objects, called here ordered fuzzy numbers, which are represented by pairs of continuous functions defined on the unit interval [0,1]. Those pairs are the counterparts of the inverses of the increasing and decreasing parts of convex fuzzy numbers. In a particular case, for the pairs (f, g) where $f, g \in C^0([0,1])$ which satisfy: 1)$f \leq g$ and 2)f and g are invertible, with f increasing and g decreasing, one can recover the class of fuzzy numbers called convex ones [3,20]. Then as long as multiplication by negative numbers is not performed, classical fuzzy calculus is equivalent to the present operations defined for ordered fuzzy numbers (with negative orientation).

Doing the present development, we would like to refer to one of the very first representations of a fuzzy set defined on a universe X (the real axis $\boldsymbol{R}$, say) of discourse. In that representation (cf. [6,26]) a fuzzy set (read here: a fuzzy number) A is defined as a set of ordered pairs $\{(x, \mu_x); x \in X\}$, where $\mu_x \in [0, 1]$ has been called the grade (or level) of membership of x in A. At that stage, no other assumptions concerning μ_x have been made. Later on, one assumed that μ_x is (or must be) a function of x. However, originally, A was just a relation in a product space $X \times [0, 1]$.

3 Attempts

A number of attempts to introduce non-standard operations on fuzzy numbers have been made [1,3,7,22,23]. It was noticed that in order to construct operations more suitable for their algorithmisation a kind of invertibility of their membership functions is required. In [10,16,17,20] the idea of modelling fuzzy numbers by means of convex or quasi-convex functions (cf. [19]) is discussed. We continue this work by defining quasi-convex functions related to fuzzy numbers in a more general fashion (called a fuzzy observation, compare its definition in [14]) enabling modelling both dynamics of changes of fuzzy membership levels and the domain of fuzzy real itself. It is worthwhile to mention here that even starting from the most popular trapezoidal membership functions, algebraic operations can lead outside this family, towards such generalised quasi-convex functions.

That more general definition enables to cope with several drawbacks. Moreover, it seems to provide a solution for other problems, like, e.g., the problem of defining an ordering over fuzzy numbers (cf.[12]). Here we should mention that Klir was the first who in [7] has revised fuzzy arithmetic to take relevant requisite constraint (the equality constraint, exactly) into account and obtained $A - A = 0$, with crisp 0, as well as the existence of inverse fuzzy numbers for the arithmetic operations. Some partial results of similar importance were obtained by Sanchez in [22] by introducing an extended operation of a very complex structure. Our approach, however, is much simpler from mathematical point of view, since it does not use the extension principle but refers to the functional representation of fuzzy numbers in a more direct way.

In the classical approach the extension principle gives a formal apparatus to carry over operations (e.g. arithmetic or algebraic) from sets to fuzzy sets. Then in the case of the so-called convex fuzzy numbers (cf. [3]) the arithmetic operations are algorithmised with the help of the so-called α-sections of membership functions. It leads to the operations on intervals. The local invertibility of quasi-concave membership functions, on the other hand, enables to define operations in terms of the inverses of the corresponding monotonic parts, as was pointed out in our previous papers [11,13]. In our last paper [14] we went further and have defined a more general class of fuzzy number, called ordered fuzzy number, just as a pair of continuous functions defined on the interval $[0, 1]$. Those pairs are counterparts of the mentioned inverses.

4 Ordered Fuzzy Numbers

Here the concept of membership functions [1] is weakened by requiring a mere *membership relation* and a fuzzy number is identified with an ordered pair of continuous real functions defined on the interval $[0, 1]$.

Definition 1. *By an ordered fuzzy number A we mean an ordered pair $A = (f, g)$ of continuous functions $f, g : [0, 1] \to \boldsymbol{R}$.*

We call the corresponding elements: f – the **up-part** and g – the **down-part** of the fuzzy number A. The continuity of both parts implies their images are bounded intervals, say

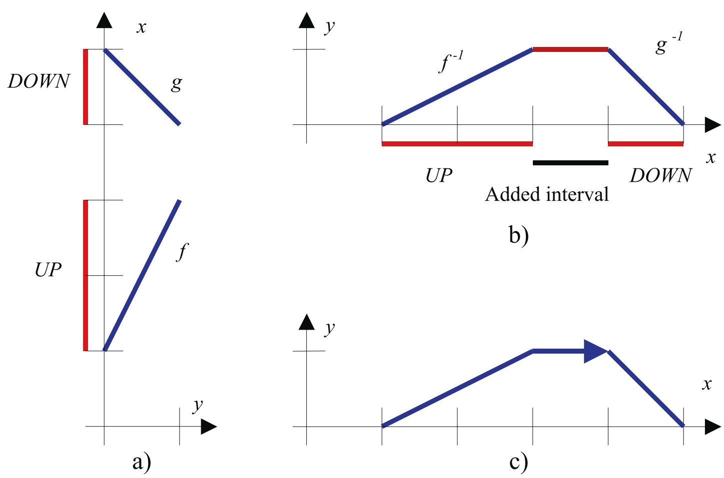

Fig. 1. a) Ordered fuzzy number, b) Ordered fuzzy number presented as fuzzy number in classical meaning, c) Simplified mark denotes the order of inverted functions

UP and $DOWN$, respectively (Fig. 1a). Let us use symbols to mark boundaries for $UP = (l_A, 1_A^+)$ and for $DOWN = (1_A^-, p_A)$.

In general, functions f, g need not be invertible, only continuity is required; they give the real variable $x \in \boldsymbol{R}$ in terms of $y \in [0, 1]$, if one refers to the classical membership function denotation. If we add a function of x on the interval $[1_A^+, 1_A^-]$ with constant value equal to 1, we may define a kind of membership function (relation) of a fuzzy set. When $f \leq g$ are both invertible, f is increasing, and g is decreasing, we get a mathematical object which presents a convex fuzzy number in the classical sense [7,23].

We can appoint an extra feature, named an orientation (marked by an arrow in Fig. 1c), to underline that we are dealing with an ordered pair of functions.

Definition 2. *Let $A = (f_A, g_A), B = (f_B, g_B)$ and $C = (f_C, g_C)$ be mathematical objects called ordered fuzzy numbers. The sum $C = A + B$, subtraction $C = A - B$, product $C = A \cdot B$, and division $C = A/B$ are defined by formula*

$$f_C(y) = f_A(y) \star f_B(y) \qquad \wedge \qquad g_C(y) = g_A(y) \star g_B(y) \tag{1}$$

where "$\star$" works for "$+$", "$-$", "$\cdot$", and "$/$", respectively, and where A/B is defined, iff zero does not belong to intervals UP and $DOWN$ of B.

Subtraction of B is the same as addition of the opposite of B, i.e. the number $(-1) \cdot B$. If for $A = (f, g)$ we define its complement $\bar{A} = (-g, -f)$ (please note that $\bar{A} \neq (-1) \cdot A$), then the sum $A + \bar{A}$ gives a fuzzy zero $0 = (f - g, -(f - g))$ in the sense of the classical fuzzy number calculus.

Definition 3. *Let $A = (f_A, g_A), B = (f_B, g_B)$ and $C = (f_C, g_C)$ be mathematical objects called ordered fuzzy numbers. The maximum $C = \max(A, B) = A \vee B$ and the minimum $C = \min(A, B) = A \wedge B$ are defined by formula*

$$f_C(y) = func\{f_A(y), f_B(y)\} \qquad \wedge \qquad g_C(y) = func\{g_A(y), g_B(y)\} \tag{2}$$

where "$func$" works for "$\max$" and "$\min$", respectively.

Many operations can be defined in this way, suitable for the pairs of functions. The Fuzzy Calculator has been already created as a calculation tool by our co-worker Mr. Roman Koleśnik [9]. It lets an easy future use of all mathematical objects described as ordered fuzzy numbers.

5 Further Extensions

Banach Algebra. The pointwise multiplication by a scalar (crisp) number, together with the operation addition lead to a *linear structure* $\mathcal{R}$ the set of all OFN's, which is isomorphic to the linear space of real 2D vector-valued functions defined on the unit interval $I = [0, 1]$. Further, one can introduce the norm over $\mathcal{R}$ as follows:

$$||x|| = \max(\sup_{s \in I} |x_{up}(s)|, \sup_{s \in I} |x_{down}(s)|) \tag{3}$$

Hence $\mathcal{R}$ can be identified with $C([0,1]) \times C([0,1])$. Finally, $\mathcal{R}$ is a Banach algebra with the unity $(1^\dagger, 1^\dagger)$– a pair of constant functions $1^\dagger(y) = 1$, for $y \in [0,1]$. Previously, a Banach structure of an extension of convex fuzzy numbers was introduced by Goetschel and Voxman [5]. However, they were only interested in the linear structure of this extension.

Order Relation and Ideals. On the space $\mathcal{R}$ we can introduce a pre–order [15] by defining a function W with the help of the relation

$$W(A) = (x_{up} + x_{down}), \tag{4}$$

its value $W(A)$ is a variation of the number $A = (x_{up}, x_{down})$. Then we say that the ordered fuzzy number A is not smaller than the number B, and write $A \succ B$, if

$$W(A) \geq W(B) \Leftrightarrow W(A - B) \geq 0 \tag{5}$$

i.e. when the function $W(A - B)$ is non–negative. We say that the number C is nonnegative if its variation is not smaller than zero, i.e. when $W(C) \geq 0$. Notice that there are ordered fuzzy numbers that are not comparable with zero. We say that a number D is around zero if its variation is the constant function equal to zero, i.e. $W(D) = 0^\dagger$. Thanks to this relation in the Banach algebra $\mathcal{R}$ we may define two ideals: the left and the right ones, which are non-trivial and possess proper divisors of zero [15].

Defuzzyfication. This is the main operation in fuzzy inference systems and fuzzy controllers [1,17,25]. The problem arises what can be done for the generalisation of classical fuzzy numbers onto ordered fuzzy numbers? In the case of the product space $\mathcal{R}$, according to the Banach-Kakutami-Riesz representation theorem, each bounded linear functional ϕ is given by a sum of two bounded, linear functionals defined on the factor space $C([0,1])$, i.e.

$$\phi(x_{up}, x_{down}) = \int_0^1 x_{up}(s)\mu_1(ds) + \int_0^1 x_{down}(s)\mu_2(ds) \tag{6}$$

where the pair of continuous functions $(x_{up}, x_{down}) \in \mathcal{R}$ represents an ordered fuzzy number, and μ_1, μ_2 are two Radon measures on $[0,1]$.

From this formula an infinite number of defuzzyfication methods can be defined. In particular, the standard procedure given in terms of the area under membership function can be generalised. It is realised by the pair of linear combinations of the Lebesgue measure of $[0,1]$. Moreover, a number of non-linear defuzzyfication operators can be defined as compositions of multivariant nonlinear functions defined on the Cartesian products of $\boldsymbol{R}$ and linear continuous functionals on the Banach space $\mathcal{R}$ [15].

It is worthwhile to mention that some further generalisations of ordered fuzzy numbers to ordered fuzzy sets (defined on a different universe than reals $\boldsymbol{R}$) can be introduced (cf. [15]). Moreover, one can think about weakening of the continuity assumption made in our fundamental definition of ordered fuzzy numbers and to consider pairs of real valued functions of the interval $[0,1]$ that are of bounded variation. Then all algebraic properties of new objects will be preserved with a small change of the norm. However, it will the subject of the next publication.

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