BULLETIN OF THE POLISH
ACADEMY OF SCIENCES
MATHEMATICS
Vol. 51, No. 3, 2003

COMPUTER SCIENCE

Ordered Fuzzy Numbers

by

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Presented by Jan RYCHLEWSKI on August 9, 2002(*)

Summary. Fuzzy counterpart of real numbers is presented. Fuzzy membership functions, which satisfy conditions similar to the quasi-convexity are considered. An extra feature, called the orientation of the curve of fuzzy membership function, is introduced. It leads to a novel concept of an ordered fuzzy number, represented by the ordered pair of real continuous functions. Arithmetic operations defined on ordered fuzzy numbers enable to avoid some drawbacks of the classical approach.

1. Introduction. In real-life problems both parameters and data used in mathematical modeling are vague. The vagueness can be described by fuzzy numbers and fuzzy sets. Fuzzy data analysis requires more than fuzzy logic: it requires fuzzy arithmetic. Its development should be based on a well established theory of fuzzy sets defined on the real axis, in order to build a fuzzy counterpart of real numbers [1], [5], [7], [12], [15]. In the classical approach for numerical handling of fuzzy quantities the so-called *extension principle* is of fundamental importance. Formulated by Zadeh [17], [18], [19], it provides a formal apparatus to carry over operations (arithmetic or algebraic) from sets to fuzzy sets.

The commonly accepted theory of fuzzy numbers is that set up by Dubois and Prade [3], who proposed a restricted class of membership functions, called (L, R)-numbers. The essence of their representation is that the membership function is of a particular form that is generated by two so-called shape (or spread) functions: L and R. In this context (L, R)-numbers became quite popular, because of their good interpretability and relatively easy handling for simple operation, i.e. for the fuzzy addition.

2000 MS Classification: primary: 68Q10; secondary: 6504, 65G30, 12K99, 93C42, 16–04, 16Z05.

Key words: fuzzy numbers, quasi-convexity, orientation, algebraic operations.

(∗) Revised version received on December 11, 2002.

If functions L and R are linear, the membership functions of fuzzy numbers become triangular. Triangular fuzzy numbers can be used for the formalization of such terms as, say *around 3* or *much more than 60.* However, approximations of fuzzy functions and operations are needed, if one wants to stay within this representation while following the Zadeh extension principle. It leads to some drawbacks concerned with the properties of fuzzy algebraic operations, as well as to unexpected and uncontrollable results of repeatedly applied operations, caused by the need of intermediate approximations [15], [16].

The goal of our paper is to overcome the above mentioned drawbacks by constructing a revised concept of the fuzzy number. It will be done first by presenting new intuitions that stay behind the final version of the definition of the concept. The new concept makes possible a simple utilizing the fuzzy arithmetic and the construction of the abelian group of fuzzy numbers. The other aim of the paper is to build foundations for constructing four operations on fuzzy numbers in a way that makes of them a field of numbers. Moreover, definitions of four main operations addition, subtraction, multiplication and subtraction have to be introduced in the form suitable for their algorithmisation.

Doing the present development, we would like to refer to one of the very first representations of a fuzzy set defined on a universe X (the real axis $\mathbb{R}$, say) of discourse, i.e. on the set of all feasible numerical values (observations, say) of a fuzzy concept (say: variable or physical measurement). In that representation (cf. [5], [17]) a fuzzy set (read here: a fuzzy number) A is defined as a set of ordered pairs $\{(x, \mu_x)\}$, where $x \in X$ and $\mu_x \in [0,1]$ has been called the grade (or level) of membership of x in A. At that stage, no other assumptions concerning μ_x have been made. Later on, one assumed that μ_x is (or must be) a function of x. However, originally, A was just a relation in a product space $X \times [0,1]$. We know that not every relation must be a functional one. It is just a commonly adopted point of view, that such a kind of relation between μ_x and x should exist, which leads to a membership function $\mu_A : X \to [0,1]$ with $\mu_x = \mu_A(x)$. In our opinion such a point of view may be too restrictive and here most of the above and further quoted problems have their origin.

The organization of the paper is the following. In Section 2 we list some drawbacks of the present fuzzy number arithmetic together with our new intuitions. In Section 3 the standard concept of a fuzzy real with the membership function, which is strictly convex, is analyzed together with the quasi-invertibility properties. In Section 4 the new definition of an ordered fuzzy number is given. In Section 5 the oriented fuzzy number is defined, which in fact forms the primitive concept for the ordered fuzzy numbers. In Section 6 the revised arithmetic operations for ordered fuzzy numbers are shortly defined and their basic properties are listed. It is evident that most of the drawbacks of the previous approaches are omitted. Fuzzy arithmetic introduced in that way has properties similar to the arithmetic of (crisp) real numbers. For sure, further studies on interpretation of the presented extended notion of a fuzzy number are necessary. However, let us claim that the standard fuzzy numbers are special cases of the ordered fuzzy numbers and we simply need that richer framework to perform basic algebraic operations without the need of intermediate approximations.

2. Intuitions. Fuzzy numbers and fuzzy arithmetic were introduced in 1975 by Zadeh [18] to analyze and manipulate approximate numerical values. In most approaches fuzzy numbers are regarded as fuzzy sets [17] defined over the real axis, that fulfill some conditions, e.g. they are normal, compactly supported and in some sense convex (cf. [2], [3], [5], [15]). Then the basic four arithmetic operations are defined on membership functions of the corresponding fuzzy sets with the help of the Zadeh's *extension principle* formulated in [18] (compare also [3]). Such an approach is represented by the majority of the researches of the subject. Unfortunately, as mentioned in Section 1, the use of the extension principle to define the arithmetic operations on fuzzy numbers results in some serious drawbacks, especially in case of performing the whole sequences of operations repeatedly (see e.g. [15], [16]).

A number of attempts to introduce non-standard operations on fuzzy numbers have been made [1], [2], [6], [14], [15]. It was noticed that in order to construct suitable operations on fuzzy numbers a kind of invertibility of their membership functions is required. In [8], [11] the idea of modeling fuzzy numbers by means of quasi-convex functions (cf. [13]) is discussed. We continue this work by defining quasi-convex functions related to fuzzy numbers in a more general fashion, enabling modeling both dynamics of changes of fuzzy membership levels and the domain of fuzzy real itself. Even starting from the most popular trapezoidal membership functions, algebraic operations can lead outside this family, towards such generalized quasi-convex functions.

That more general definition enables to cope with the main drawback of other approaches, namely that the difference $A - A$ is usually a fuzzy zero-not the crisp zero. Moreover, it seems to provide a solution for other problems like e.g. the problem of defining total ordering over fuzzy numbers (cf. [7]). Here we should mention that Klir was the first, who in [6] has revised fuzzy arithmetic to take relevant requisite constraint (the equality constraint, exactly) into account and obtained $A - A = 0$ as well as the existence of inverse fuzzy numbers for the arithmetic operations. Some partial results of the similar importance were obtained by Sanchez in [14] by introducing an extended operation of a very complex structure. Our approach, however, is much simpler from mathematical point of view, since it does not use the extension principle but refers to the functional representation of fuzzy numbers in a more direct way.

We would like here to form new intuitions concerning the fuzzy numbers. In our opinion, the existence of the membership function is from one side a very convenient fact as far as a simple interpretation in the set-theoretical language is concerned. However, on the other side, it implies an extra restrictions. Operations on real numbers were introduced several 1000's years ago without any correspondence to characteristic functions of one-element sets. First, the operation of addition had been introduced between natural numbers. Integers appeared relatively late, when a need for the representation of subtraction occurred. Hence, the human being was able to solve a simple equation $a + x = c$ uniquely, with a and c natural. If we now come back to the fuzzy case and start with the subset $\mathcal{FN}^+$ - the subset of fuzzy numbers corresponding to quasi-convex membership functions supported within $\mathbb{R}^+$ (possessing a more advanced algebraic structure) - by asking for a solution of the equation $A + X = C$, the answer is unknown, in general. We are not satisfied with that and we could ask: What is the fuzzy counterpart of the difference $c - a$, if a and c are natural numbers? Can we find any information, say X, in such a way that by adding it to the specified quasi-convex fuzzy number A we obtain the specified quasi-convex fuzzy number C? In our previous papers [9, 10] we have already tried to answer these questions in terms of the so-called *improper parts* of fuzzy numbers. Although by letting such parts we cannot keep such a clear interpretation like e.g. that developed for fuzzy numbers within the possibility theory (cf. [4]), we obtain a very simple (one could say: pragmatic) way of revision of fuzzy information. Namely, given "proper" fuzzy numbers A and C, we can always specify possibly "improper" X such that $A + X = C$. Moreover, as shown in the foregoing sections, we can do it in a very easy (one could say: obvious) way, much clearer and efficient than in case of other approaches.

In our previous paper [9], [10], we claimed that an extra feature of fuzzy number should be added in order to distinguish between two kinds of fuzzy numbers: mirror images of positive numbers $A \in \mathcal{FN}^+$ and negative numbers $B \in \mathcal{FN}^-$ defined diretcly on $\mathbb{R}^-$. To solve this problem we have introduced the concept of the *orientation of the membership curve* of A. To be more evident, let us assume for a while that a prototype of our fuzzy number is a fuzzy set with its support containing a finite number of points, $x_1, x_2, x_3, x_4 \in \mathbb{R}$ with the corresponding levels of membership $y_1, y_2, y_3, y_4 \in (0, 1]$. Hence, we will distinguish two ordered sets (lists, exactly):

$$A = \{(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)\}$$
$$C = \{(x_4, y_4), (x_3, y_3), (x_2, y_2), (x_1, y_1)\} \tag{1}$$

and say that the orientation of the fuzzy set C is opposite to the orientation of A. If we assume that all $x_i, i = 1, 2, 3, 4$, are positive, then the mirror image of A, denoted by $-A$, is

$$-A = \{(-x_4, y_4), (-x_3, y_3), (-x_2, y_2), (-x_1, y_1)\} \tag{2}$$

and it is obvious now that $-A$ is different from the set

$$B = \{(-x_1, y_1), (-x_2, y_2), (-x_3, y_3), (-x_4, y_4)\} \tag{3}$$

Using the different language, we can say that the observation leading to the set C was made in the opposite direction to the observation leading to A. In Section 4 we will give a precise definition of the fuzzy observation that leads to the definition of ordered fuzzy number and then to the revised arithmetic.

3. Fuzzy numbers. The objective of considering fuzzy numbers is to express the vagueness concerned with the observed real values. Let $r \in \mathbb{R}$. Its representation in the set-theoretical language is characteristic function of the one-element set $\{r\}$:

$$\chi_r(x) = \begin{cases} 1 & \text{if } x = r \\ 0 & \text{if } x \neq r. \end{cases} \tag{4}$$

By a fuzzy real approximating the above χ_r one can mean $A = (\mathbb{R}, \mu_A)$, where $\mu_A : \mathbb{R} \to [0, 1]$ assigns positive values to numbers in neighbourhood of r, with $\mu_A(r) = 1$. Function μ_A is called the **membership function** of fuzzy number A. In [8] conditions distinguishing fuzzy membership functions corresponding to fuzzy numbers are proposed. They are in some sense weaker than those proposed by Dubois and Prade [3], more similar to those given by Wagenknecht in [15]. The crucial point is the assumption of quasi-invertibility. Let us recall that notion by following [13]:

Definition 1. Function $f : \mathbb{R} \to \mathbb{R}$ is **strictly quasi-convex**, iff for any $x, y, z \in \mathbb{R}$, $x < y < z$, we have

$$f(x) \leqslant f(z) \Rightarrow f(y) \leqslant f(z) \quad \text{and} \quad f(x) < f(z) \Rightarrow f(y) < f(z) \tag{5}$$

Let us specify two types of fuzzy numbers $A = (\mathbb{R}, \mu_A)$:

— **crisp**, where $\mu_A = \chi_r$, $r \in \mathbb{R}$.

— **genuinely fuzzy**, where (1) μ_A is normal, i.e. $1 \in \mu_A(\mathbb{R})$, (2) the support of μ_A is an interval (l_A, p_A), $l_A, p_A \in \mathbb{R}$, (3) $-\mu_A$ is strictly quasi-convex.

According to the fundamental theorem for strictly quasi-convex functions [14], the above genuinely fuzzy numbers are quasi-invertible:

Proposition 1 (cf. [13]). *Let genuinely fuzzy* $A = (\mathbb{R}, \mu_A)$ *be given. Then there exist* $1_A^-, 1_A^+ \in (l_A, p_A)$ *such that* μ_A *is invertible (increasing) on* $(l_A, 1_A^-)$*, invertible (decreasing) on* $(1_A^+, p_A)$*, and constantly equal to 1 on* $[1_A^-, 1_A^+]$.

Quasi-invertibility enables to state quite an efficient calculus on fuzzy numbers [8]. Given $A = (\mathbb{R}, \mu_A)$, $B = (\mathbb{R}, \mu_B)$, one can construct the sum $A + B$ by pairwise adding inverses of the increasing and decreasing parts of functions μ_A and μ_B. In case of trapezoidal fuzzy membership functions, this operation can be encoded in terms of equations $l_C = l_A + l_B$, $1_C^- = 1_A^- + 1_B^-$, $1_C^+ = 1_A^+ + 1_B^+$ and $p_C = p_A + p_B$ (cf. Fig. 1). Analogously, one could define subtraction $A - B$. However, as a result we can obtain a plot, which does not correspond to a function any more (cf. Fig. 2). In [9], [10] such a result has been called the improper part of fuzzy number. Another problem is that it is impossible to follow with the notion of a reverse fuzzy number $-A$ to get crisp zero as the result of $A - A$ (cf. Fig. 3).

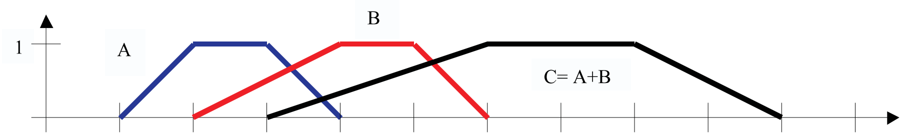

Fig. 1. Addition of two fuzzy numbers, $C = A + B$

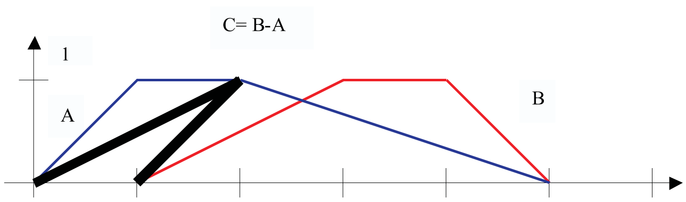

Fig. 2. Subtraction of two fuzzy numbers, $C = B - A$

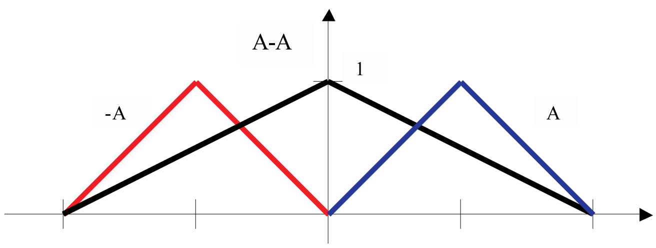

Fig. 3. Operation $A - A$ for an exemplary fuzzy number A

4. Fuzzy observations and ordered fuzzy numbers. In [11] the idea of treating fuzzy numbers as pairs of functions defined on interval $(0,1]$ has occurred. Those two functions, introduced with the help of α-sections, played the role of generalized inverses of monotonic parts of fuzzy membership functions. We are following this idea by providing much clearer, novel interpretation of fuzzy numbers. We obtain a richer generalization of fuzzy numbers, which seems to be more relevant to the needs of applications. Let us begin with the following characteristics, a simple modification of Proposition 1:

PROPOSITION 2. *Let continuous function $\boldsymbol{f} : \mathbb{R} \to \mathbb{R} \times [0,1]$ be given. For any $t \in \mathbb{R}$ let us write $\boldsymbol{f}(t) = (x_f(t), \mu_f(t))$. It satisfies the following properties*

1. $\{t \in \mathbb{R} : \mu_f(t) > 0\} = (t_0, t_1)$, *for some* $t_0, t_1 \in \mathbb{R}$
2. $\mu_f(t) = 1$ *for some* $t \in (t_0, t_1)$
3. $-\mu_f|[t_0, t_1] : \mathbb{R} \to [-1, 0]$ *is strictly quasi-convex*

iff there exist $t^-, t^+ \in (t_0, t_1)$ *such that* μ_f *is constantly equal to* 1 *on* $[t^-, t^+]$, *as well as increasing on* $[t_0, t^-]$ *and decreasing on* $[t^+, t_1]$, *i.e. iff there exist continuous functions* $\mu_f^{\uparrow}, \mu_f^{\downarrow} : [0,1] \to \mathbb{R}$ *such that*:

$$\begin{aligned} &\boldsymbol{f}(t_0) = (\mu_f^{\uparrow}(0), 0) \qquad \boldsymbol{f}(t^-) = (\mu_f^{\uparrow}(1), 1) \\ &\boldsymbol{f}(t_1) = (\mu_f^{\downarrow}(0), 0) \qquad \boldsymbol{f}(t^+) = (\mu_f^{\downarrow}(1), 1) \end{aligned} \tag{6}$$

and:

$$\begin{aligned} &\forall_{t \in (t_0, t^-)} \exists!_{\, y \in (0,1)} \boldsymbol{f}(t) = (\mu_f^{\uparrow}(y), y) \\ &\forall_{t \in (t^+, t_1)} \exists!_{\, y \in (0,1)} \boldsymbol{f}(t) = (\mu_f^{\downarrow}(y), y) \end{aligned} \tag{7}$$

Function $\boldsymbol{f}$ can be redefined as $\boldsymbol{f} : [t_0, t_1] \to [x_0, x_1] \times [0,1]$, where $\boldsymbol{f}(t_0) = (x_0, 0)$ and $\boldsymbol{f}(t_1) = (x_1, 0)$. Values $t_0, t_1 \in \mathbb{R}$ can be interpreted as the beginning and the ending time of some physical experiment, during which we are remembering the observed outcomes together with the degrees of their membership to the considered fuzzy concept. Let us call $\boldsymbol{f}$ a *fuzzy observation.* According to Proposition 2, function $\boldsymbol{f}$ can be identified with the ordered pair $(\mu_f^{\uparrow}, \mu_f^{\downarrow})$. Let us use it as follows:

DEFINITION 2. By *ordered fuzzy real* we mean ordered pair $A = (\mu_A^{\uparrow}, \mu_A^{\downarrow})$, where $\mu_A^{\uparrow}, \mu_A^{\downarrow} : [0,1] \to \mathbb{R}$ are continuous functions. We call the corresponding (due to Proposition 2) function $\boldsymbol{f} : \mathbb{R} \to \mathbb{R} \times [0,1]$ a *fuzzy A-observation*, denoted by $\boldsymbol{f}_A = (x_A, \mu_A)$, where $x_A : \mathbb{R} \to \mathbb{R}$ and $\mu_A : \mathbb{R} \to [0,1]$.

One can see that by defining $x_A : \mathbb{R} \to \mathbb{R}$ as $x_A(t) = t$ or $x_A(t) = -t$ we obtain the classical notion of fuzzy number. Functions $(\mu_A^{\uparrow}, \mu_A^{\downarrow})$ correspond then to inverses of monotonic parts of μ_A, as understood in Proposition 1. Let us consider two fuzzy observations $\boldsymbol{f}, \boldsymbol{g}$ being opposite to each other, in terms of equations

$$\forall_{t\in\mathbb{R}}\, [x_f(t) = -x_g(t) \;\wedge\; \mu_f(t) = \mu_g(t)] \tag{8}$$

stating that the same fuzzy membership coefficients are observed in a completely opposite way. We claim that such a correspondence leads to a sound definition of a reverse fuzzy number. More specifically, we refer to ordered fuzzy numbers, by basing on the following result:

Proposition 3. *Let fuzzy observations* $\boldsymbol{f}, \boldsymbol{g} : \mathbb{R} \to \mathbb{R} \times [0,1]$ *be given. Let us consider ordered pairs* $(\mu_f^{\uparrow}, \mu_f^{\downarrow})$, $(\mu_g^{\uparrow}, \mu_g^{\downarrow})$ *corresponding to* $\boldsymbol{f}$ *and* $\boldsymbol{g}$, *provided by Proposition* 2. *If equation* (8) *is satisfied, then one has also*

$$\forall_{y\in[0,1]} \left[\mu_f^{\uparrow}(y) = -\mu_g^{\uparrow}(y) \;\wedge\; \mu_f^{\downarrow}(y) = -\mu_g^{\downarrow}(y)\right] \tag{9}$$

5. Oriented fuzzy numbers. Let us go back for a while to the classical notion of a fuzzy real. Consider a piecewise continuous and nonnegative function μ_A defined on $[a,b]$, where $a \leqslant b$, $\mu_A(a) = \mu_A(b) = 0$. If in $\mathbb{R}^2$ one adds segment $[a,b] \times \{0\}$ to the plot of μ_A, then the resulting set

$$\mathcal{C}_A = \{(x,y) \in \mathbb{R}^2 : y = \mu_A(x), x \in [a,b]\} \cup [a,b] \times \{0\} \tag{10}$$

can be treated as a plane closed curve. Such a curve can possess two orientations: *positive* - if together with the increasing parameter of the curve its running point moves counterclockwise (compatible with the orientation of the coordinate vectors - see Fig. 4), and *negative* - in the opposite case.

Definition 3. By *oriented fuzzy number* we mean triple $A = (\mathbb{R}, \mu_A, \mathsf{s}_A)$, where $\mu_A : \mathbb{R} \to [0,1]$ represents either crisp or genuinely fuzzy number, and $\mathsf{s}_A \in \{-1, 0, 1\}$ denotes orientation of A.

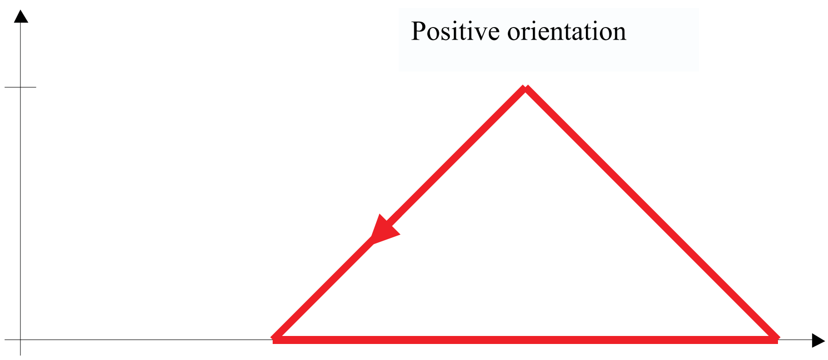

Fig. 4. Positively oriented fuzzy real

We distinguish two types of oriented fuzzy numbers. For crisp A we have $\mathsf{s}_A = 0$, while for genuinely fuzzy A, there is $\mathsf{s}_A = 1$ or $\mathsf{s}_A = -1$, depending on the orientation of curve $\mathcal{C}_A$. The notion of a fuzzy number can be redefined in terms of tuple $A = (\mathcal{C}_A, \mathsf{s}_A)$. Consider closed curve $\mathcal{C}_A \subseteq \mathbb{R}^2$ written as

$$\mathcal{C}_A = [l_A, p_A] \times \{0\} \cup [1_A^-, 1_A^+] \times \{1\} \cup \mathrm{up}_A \cup \mathrm{down}_A \tag{11}$$

where $\mathrm{up}_A, \mathrm{down}_A \subseteq \mathbb{R}^2$ are the plots of some monotonic functions . (To have a common approach to crisp and genuinely fuzzy numbers in terms of membership curves, we are adding the interval $[0,1] \times r$ to the graphical representation of a crisp number χ_r .) Abbreviations up and down do not correspond to the fact whether a given function is increasing or decreasing but to the orientation of $\mathcal{C}_A$. They simply label the ascending and descending parts of the oriented curve. We call such interpreted subsets $\mathcal{C}_A \subseteq \mathbb{R}^2$ *membership curves* of oriented fuzzy numbers.

Membership curves provide graphical representation of ordered fuzzy numbers as well. Functions $\mu_A^{\uparrow}$ and $\mu_A^{\downarrow}$ correspond to $\mathrm{up}_A, \mathrm{down}_A \subseteq \mathbb{R}^2$ as follows:

$$\mathrm{up}_A = \{(\mu_A^{\uparrow}(y), y) : y \in [0,1]\} \qquad \mathrm{down}_A = \{(\mu_A^{\downarrow}(y), y) : y \in [0,1]\} \tag{12}$$

In the language of fuzzy observations $\boldsymbol{f}_A = (x_A, \mu_A)$, one can notice that:

1. Observations $\boldsymbol{f}_A$ with increasing x_A acquire orientation $\mathsf{s}_A = -1$.
2. Observations $\boldsymbol{f}_A$ with decreasing x_A acquire orientation $\mathsf{s}_A = 1$.
3. Observations $\boldsymbol{f}_A$ with constant x_A acquire orientation $\mathsf{s}_A = 0$.

If we let up_A, $\mathrm{down}_A \subseteq \mathbb{R}^2$ correspond to not necessarily invertible functions from $[0,1]$ to $\mathbb{R}$, then we obtain equivalence between ordered and oriented fuzzy numbers. We are going to use the curves of oriented numbers to illustrate operations performed on the ordered ones. We simply state that orientation of $\mathcal{C}_A$ corresponds to the way of situating the plots of $\mu_A^{\uparrow}$ and $\mu_A^{\downarrow}$ at the plane.

As an example, let us consider Fig. 5, which illustrates intuition of defining the reverse of ordered (oriented) fuzzy real. According to the idea mentioned at the end of previous section (formalized partially by means of Proposition 3), such a reverse should obtain an opposite orientation. By performing operation of addition on the pairs of ascending and descending parts of A and $-A$, we are then able to obtain crisp zero.

6. Arithmetic operations on ordered fuzzy numbers. Let us formulate operations of addition and subtraction on ordered fuzzy numbers. The following definition coincides with that presented for oriented fuzzy numbers in [9] and [10]. It is also related to the idea of adding and subtracting fuzzy numbers by means of inverses of their monotonic parts [11].

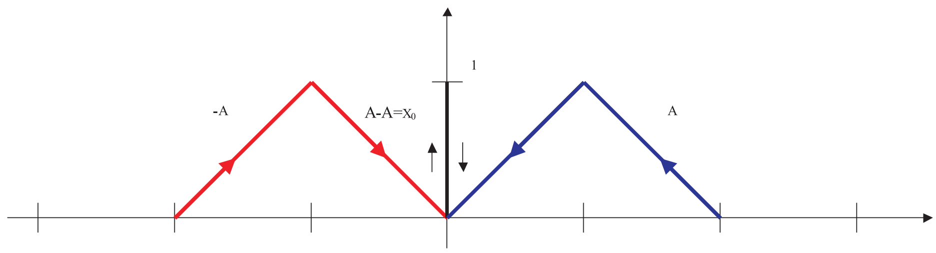

Fig. 5. Addition of (intuitively) reverse oriented (ordered) fuzzy numbers

However, characteristics of the ordered fuzzy number enables much clearer definition.

DEFINITION 4. Let three ordered fuzzy numbers, $A = (\mu_A^{\uparrow}, \mu_A^{\downarrow})$, $B = (\mu_B^{\uparrow}, \mu_B^{\downarrow})$ and $C = (\mu_C^{\uparrow}, \mu_C^{\downarrow})$, be given. We say that:

1. C is the sum of A and B, denoted by $C = A + B$, iff

$$\forall_{y\in[0,1]} \left[\mu_A^{\uparrow}(y) + \mu_B^{\uparrow}(y) = \mu_C^{\uparrow}(y) \ \wedge \ \mu_A^{\downarrow}(y) + \mu_B^{\downarrow}(y) = \mu_C^{\downarrow}(y)\right] \tag{13}$$

2. C is the subtraction of B from A, denoted by $C = A - B$, iff

$$\forall_{y\in[0,1]} \left[\mu_A^{\uparrow}(y) - \mu_B^{\uparrow}(y) = \mu_C^{\uparrow}(y) \ \wedge \ \mu_A^{\downarrow}(y) - \mu_B^{\downarrow}(y) = \mu_C^{\downarrow}(y)\right] \tag{14}$$

If $x_A(t) = -t$, then $\mu_A^{\uparrow}$ and $\mu_A^{\downarrow}$ are invertible functions of variable x. Inverses

$$(\mu_A^{\uparrow})^{-1} = \mu_A|(l_A, 1_A^-) \qquad (\mu_A^{\downarrow})^{-1} = \mu_A|(1_A^+, p_A) \tag{15}$$

form decreasing and increasing parts of membership function μ_A, respectively. The result of addition of numbers A and B, where $x_A(t) = x_B(t) = -t$, is defined exactly as in case of convex fuzzy numbers, for which the interval arithmetic is used (cf. [2], [8]). Such numbers obtain positive orientation, which is also the case of their sum (see Fig. 6).

Subtraction of positively oriented fuzzy numbers is an ordered (oriented) fuzzy real, but may not be a classical fuzzy real. Still, in some cases, the outcome of subtraction can be interpreted in classical terms, as in Fig. 7. Subtraction of A from A is the same operation as addition to A its reverse, i.e. the number $-A = (-\mu_A^{\uparrow}, -\mu_A^{\downarrow})$. Then we get $C = (\mu_C^{\uparrow}, \mu_C^{\downarrow})$, where

$$\mu_C^{\uparrow}(y) = \mu_A^{\uparrow}(y) - \mu_A^{\uparrow}(y) = 0 \qquad \mu_C^{\downarrow}(y) = \mu_A^{\downarrow}(y) - \mu_A^{\downarrow}(y) = 0 \tag{16}$$

Since the crisp number $r \in \mathbb{R}$ (in other words - characteristic function χ_r) can be represented by the ordered pair $(\mu_r^{\uparrow}, \mu_r^{\downarrow})$, where

$$\forall_{y\in[0,1]} \left[\mu_r^{\uparrow}(y) = \mu_r^{\downarrow}(y) = r\right] \tag{17}$$

the result of such operation is just the crisp number $r = 0$, like in Fig. 5. In general, one can expect the results not necessarily interpretable in terms of classical approach but still valuable as representing some fuzzy membership information, like in Fig. 8. Obviously, this is not a problem at the level of ordered fuzzy numbers, where orientation of the membership curve just illustrates the position of $\mu_A^{\uparrow}$ with respect to $\mu_A^{\downarrow}$. Actually, it will not be a problem to extend the arithmetical framework of ordered fuzzy numbers by multiplication and appropriately defined division, which will be the next step in our research.

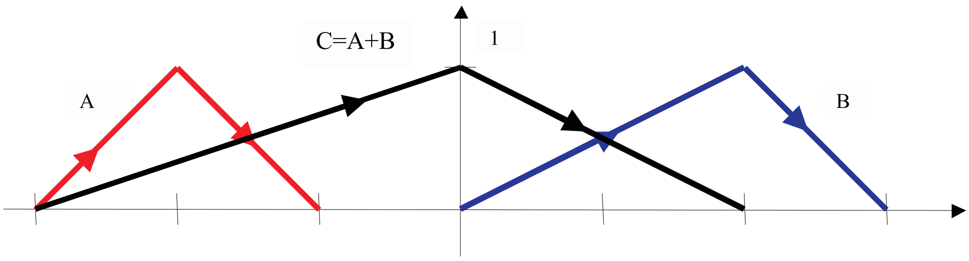

Fig. 6. Addition of ordered, positively oriented fuzzy numbers

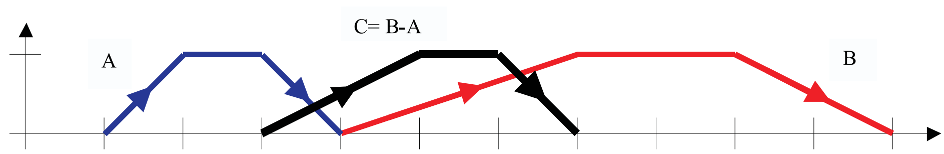

Fig. 7. Subtraction resulting with a positively oriented fuzzy real

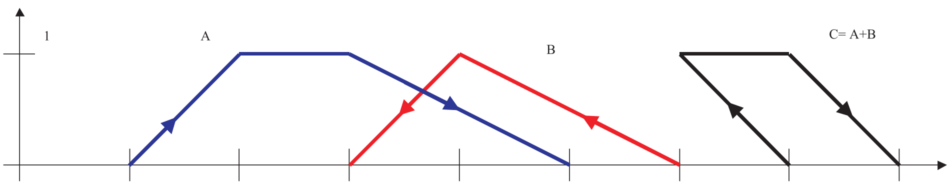

Fig. 8. Addition of ordered fuzzy numbers with opposite orientation

7. Conclusions. We introduced the notion of a fuzzy observation and an ordered fuzzy real and explained its correspondence with fuzzy numbers defined by quasi-convex membership functions, as well as with oriented fuzzy numbers known from our previous research. We stated definitions of operations of addition and subtraction on ordered fuzzy numbers and provided examples of their results.

Acknowledgements. The work on the paper has been supported by the Research Center of PJIIT and the research project of IMSiIS of the University of Bydgoszcz, both partially supported by Polish National Committee for Scientific Research (KBN). The authors thank to Professor Jacek Łęski from the Technical University of Silesia for his valuable remarks and interest in the research.

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