Introduction to Combinatorics: Basic Counting Techniques

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PhD STUDIES

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Literature

- Combinatorics:
  - D.Knuth et al. "Concrete Mathematics" (also available in Polish, PWN 1998)
  - M.Libura et al. "Wykłady z Matematyki Dyskretnej", cz.1 Kombinatoryka, skrypt WSISIZ, Warszawa 2001
  - M.Lipski "Kombinatoryka dla programistów", WNT 2004
  - Van Lint et al. "A Course in Combinatorics", Cambridge 2001
- Graphs:
  - R.Wilson "Introduction to Graph Theory" (also available in Polish, PWN 2000)
  - R.Diestel "Graph Theory", Springer 2000

Contents

General Basic Ideas for Counting

- create easy-to-count representations of counted objects
- "product rule": multiply when choices are independent
- "sum rule": sum up exclusive alternatives
- "combinatorial interpretation" proving technique

These ideas can be used not only to count objects but also to **easily prove** non-trivial discrete-maths identities (examples soon)

Counting Basic Objects

Denotations: "#" means: "the number of"

$[n]$ means: the set {1,...,n}, for $n \in \mathcal{N}$

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- (# subsets of an n-element set) $|P([n])| =$

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- # ordered placings of m different balls into n different boxes :

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- (# permutations of [n]) $|S_n| = n!$

Binomial Coefficient

k-subsets of [n]

$$\binom{n}{k} = \frac{n^{\underline{k}}}{k!}$$

$\mathcal{N} \ni k \leq n \in \mathcal{N}$
(there is also possible a more general formulation for non-natural numbers)

Recursive formulation

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

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how to prove it without (almost) any algebra?
(use "combinatorial interpretation" idea)
(consider whether one fixed element of [n] belongs to the k-subset or not)

Basic properties

Symmetry: $\binom{n}{k} = \binom{n}{n-k}$

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(each k-subset defines (n-k)-subset – its complement)

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(summing subsets of [n] by their multiplicity)

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$$\binom{m+n}{k} = \sum_{s=0}^{k}\binom{m}{s}\binom{n}{k-s}$$

Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

(explanation: each term on the right may be represented as a k-subset of [n])

Corollaries:

- # odd subsets of [n] equals # even subsets

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- # odd subsets of [n] equals # even subsets (substitute x=1 and y=-1)
- $$\sum_{k=0}^{n} k \binom{n}{k} = n2^{n-1}$$

  (take derivative (as function of x) of both sides and then subsitute x=y=1)

Multinomial Coefficient

(a generalisation of the binomial coefficient)

colourings of n balls with at most m different colours so that exactly $k_i \in N$ balls have the i-th colour

$$\binom{n}{k_1 k_2 \dots k_m} = \frac{n!}{k_1! \cdot \ldots \cdot k_m!}$$

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(fix the sequence $(k)_i$ and fix a permutation of [n] to be coloured)

Multinomial Theorem

$$(x_1+\ldots+x_m)^n = \sum_{\substack{k_1,\ldots,k_m \in \mathcal{N} \\ k_1+\ldots+k_m=n}} \binom{n}{k_1 k_2 \ldots k_m} x_1^{k_1} \cdot \ldots \cdot x_m^{k_m}$$

Corollary:

$$\sum_{\substack{k_1,\ldots,k_m \in \mathcal{N} \\ k_1+\ldots+k_m=n}} \binom{n}{k_1 k_2 \ldots k_m} = m^n$$

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(substitute all 1's on the left-hand side)

Recursive Formula

$$\binom{n}{k_1\, k_2 \ldots k_m} = \binom{n-1}{k_1 - 1\; k_2 \ldots k_m} + \binom{n-1}{k_1\; k_2 - 1 \ldots k_m} +$$

$$+ \ldots + \binom{n-1}{k_1\; k_2 \ldots k_m - 1}$$

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(explanation: fix one element of $[n]$ and analogously to the binomial theorem)

Multisets

Element of type x_i has k_i (identical) copies.

$$Q =< k_1 * x_1, \ldots, k_n * x_n >$$

$$|Q| = k_1 + \ldots + k_n$$

Subset S of Q:
$S =< m_1 * x_1, \ldots, m_n * x_n >$
$(0 \leq m_i \leq k_i$, for $i \in [n])$

Fact:
subsets of Q = ?

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$(0 \leq m_i \leq k_i, \text{ for } i \in [n])$

Fact:
subsets of Q = ? $(1 + k_1) \cdot (1 + k_2) \cdot \ldots \cdot (1 + k_n)$

Partitions of number n into sum of k terms

$$P(n, k)$$

k-partitions of n, $n \in \mathcal{N}$

Partition of n: $n = a_1 + \ldots + a_k$, $a_1 \geq \ldots \geq a_k > 0$

Recursive formula:

$$P(n, k) = P(n-1, k-1) + P(n-k, k)$$

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(decrease by 1 each of i terms greater than 1)

Set Partitions

Partition of finite set X into k blocks: $\Pi_k = A_1, \ldots, A_k$ so that:

- $\forall_{1 \leq i \leq k} A_i \neq \emptyset$
- $A_1 \cup \ldots \cup A_k = X$
- $\forall_{1 \leq i < j \leq k} A_i \cap A_j = \emptyset$

Stirling Number of the 2nd kind

$$\left\{ \begin{array}{c} n \\ k \end{array} \right\} = |\Pi_k([n])|$$

(# partitions of $[n]$ into k blocks)
Recursive formula:

$$\left\{ \begin{array}{c} n \\ n \end{array} \right\} = 1 \quad \left\{ \begin{array}{c} n \\ 0 \end{array} \right\} = 0 \textit{ for } n > 0$$

$$\left\{ \begin{array}{c} n \\ k \end{array} \right\} = \left\{ \begin{array}{c} n-1 \\ k-1 \end{array} \right\} + k \left\{ \begin{array}{c} n-1 \\ k \end{array} \right\}$$

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(a fixed element constitutes a singleton-block or belongs to one of bigger k blocks)

Equivalence Relations and Surjections

- Bell Number

$$B_n = \sum_{k=0}^{n} \left\{ \begin{array}{c} n \\ k \end{array} \right\}$$

  (# equivalence relations on [n])

- # surjections from $[m]$ onto $[n]$, $m \geq n$ = ?

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- # surjections from $[m]$ onto $[n]$, $m \geq n$ = ?

  $$|Sur([m],[n])| = m! \cdot \left\{ \begin{matrix} m \\ n \end{matrix} \right\}$$

Relation between $x^n, x^{\underline{n}}, x^{\overline{n}}$ (with Stirling 2-nd order numbers)

$$x^n = \sum_{k=0}^{n} \left\{ \begin{array}{c} n \\ k \end{array} \right\} \cdot x^{\underline{n}} = \sum_{k=0}^{n} (-1)^{n-k} \cdot \left\{ \begin{array}{c} n \\ k \end{array} \right\} \cdot x^{\overline{n}}$$

Permutations

$$Inj([n],[n])$$

Permutations on $[n]$ constitute a group denoted by S_n

- composition of 2 permutations gives a permutation on $[n]$
- identity permutation is a neutral element (e)
- inverse of permutation f is a permutation f^{-1} on $[n]$

Examples

$$f = \begin{pmatrix} 12345 \\ 53214 \end{pmatrix}$$

$$g = \begin{pmatrix} 12345 \\ 25314 \end{pmatrix}$$

Inverse:

$$f^{-1} = \begin{pmatrix} 12345 \\ 43251 \end{pmatrix}$$

The group is not commutative: fg is different than gf:

$$fg = \begin{pmatrix} 12345 \\ 34251 \end{pmatrix}$$

$$gf = \begin{pmatrix} 12345 \\ 43521 \end{pmatrix}$$

Decomposition into Cycles

A cycle is a special kind of permutation.

Each permutation can be decomposed into disjoint cycles:

- decomposition is unique
- the cycles are commutative

Example:

$$f = \begin{pmatrix} 12345 \\ 53214 \end{pmatrix}$$

$$f = f'f'' = [1,5,4][2,3] = [2,3][1,5,4] = f''f' = f$$

Stirling Number of the 1st kind

permutations consisting of exactly k cycles: $\left[\begin{matrix} n \\ k \end{matrix} \right]$

$$\sum_{k=0}^{n} \left[\begin{matrix} n \\ k \end{matrix} \right] = n!$$

Recursive Formula:

$$\left[\begin{matrix} n \\ k \end{matrix} \right] = \left[\begin{matrix} n-1 \\ k-1 \end{matrix} \right] + (n-1) \left[\begin{matrix} n-1 \\ k \end{matrix} \right]$$

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(fix a single element: it either itself constitutes a 1-cycle or can be at one of the n-1 positions in the k cycles)

Expressing decreasing and increasing powers in terms of "normal" powers with the help of Stirling Numbers of the 1-st kind

$$x^{\underline{n}} = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right] (-1)^{n-k} x^k$$

$$x^{\overline{n}} = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right] x^k$$

(Reminder of the opposite)

$$x^n = \sum_{k=0}^{n} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} \cdot x^{\underline{n}} = \sum_{k=0}^{n} (-1)^{n-k} \cdot \left\{ \begin{matrix} n \\ k \end{matrix} \right\} \cdot x^{\overline{n}}$$

Other (amazing) connections between Stirling Numbers of both kinds

$$\sum_{k=0}^{n}(-1)^{n-k}\begin{bmatrix} n \\ k \end{bmatrix}\begin{Bmatrix} k \\ m \end{Bmatrix} = [m == n]$$

$$\sum_{k=0}^{n}(-1)^{n-k}\begin{Bmatrix} n \\ k \end{Bmatrix}\begin{bmatrix} k \\ m \end{bmatrix} = [m == n]$$

Type of permutation

Permutation $f \in S_n$ has *type* $(\lambda_1, \ldots, \lambda_n)$ iff its decomposition into disjoint cycles contains exactly λ_i cycles of length i.

Example:

$$f = \begin{pmatrix} 123456789 \\ 751423698 \end{pmatrix}$$

$f = [1, 7, 6, 3], [2, 5], [4], [8, 9]$

Thus, the type of f is $(1, 2, 0, 1, 0, 0, 0, 0, 0)$.

We equivalently denote it as: $1^1 2^2 4^1$

Inversion in a permutation

Inversion in a permutation $f = (a_1, \ldots, a_n) \in S_n$ is a pair (a_i, a_j) so that $i < j \leq n$ and $a_i > a_j$

The number of inversions in permutation f is denoted as $I(f)$

What is the minimum/maximum value of $I(f)$?
How does it relate to sorting?
How to efficiently compute $I(f)$?

Transpositions

A permutation that is a cycle of length 2 is called *transposition*

Fact: Each permutation f is a composition of exactly $I(f)$ transpositions of neighbouring elements

Sign of Permutation

assume, $f \in S_n$:
(definition)

$$sgn(f) = (-1)^{I(f)}$$

$$sgn(fg) = sgn(f) \cdot sgn(g)$$

$$sgn(f^{-1}) = sgn(f)$$

We say that a permutation is *even* when $sgn(f) = 1$ and *odd* otherwise

Fact: $sgn(f) = (-1)^{k-1}$ for any f being a k-cycle

Computing the sign of a permutation

For any permutation $f \in S_n$ of the type $(1^{\lambda_1} \dots n^{\lambda_n})$ its sign can be computed as follows:

$$sgn(f) = (-1)^{\sum_{j=1}^{\lfloor n/2 \rfloor} \lambda_{2j}}$$

(only even cycles contribute to the sign of the permutation)

General Techniques

- Pigeonhole Principle
- Inclusion-Exclusion Principle
- Generating Functions

Pigeonhole Principle (Pol. "Zasada szufladkowa")

Let X,Y be finite sets, $f \in Fun(X, Y)$ and $|X| > r \cdot |Y|$ for some $r \in \mathcal{R}_{+}$. Then, for at least one $y \in Y$, $|f^{-1}(\{y\})| > r$.

(or equivalently: if you put m balls into n boxes then at least one box contains not less than m/n balls)

Examples

Any 10-subset of [50] contains two different 5-subsets that have the same sum of elements.

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("Hair strands theorem", etc.):
At the moment, there exist two people on the earth that have exactly the same number of hair strands

Inclusion-Exclusion Principle

(Pol. "Zasada Włączeń-Wyłączeń")

For any non-empty family $\mathcal{A} = \{A_1, \ldots, A_n\}$ of subsets of a finite set X, the following holds:

$$|\bigcup_{i=1}^{n} A_i| = \sum_{i=1}^{n} (-1)^{i-1} \sum_{1 \leq p_1 < p_2 < \ldots < p_i \leq n} |A_{p_1} \cap A_{p_2} \cap \ldots \cap A_{p_i}|$$

(proof by induction on n)

Example: the principle can be used to prove that

$$\left\{ \begin{array}{c} n \\ k \end{array} \right\} = \frac{1}{m!} \sum_{i=0}^{m-1} (-1)^i \left(\begin{array}{c} m \\ i \end{array} \right) (m-i)^n$$

(a closed-form formula for Stirling number of the 2-nd kind)

Example: number of Derangements

A *derangement* (Pol. "nieporządek") is a permutation $f \in S_n$ so that $f(i) \neq i$, for $i \in [n]$.

D_n is the set of all derangements on $[n]$. $|D_n|$ is denoted as $!n$.

Theorem: $!n = |D_n| = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!}$

Proof of the formula for !n

Let $A_i = \{f \in S_n : f(i) = i\}$, for $i \in [n]$. Thus
$!n = |S_n| - |A_1 \cup A_2 \cup \ldots \cup A_n| =$

$$= n! - \sum_{i=1}^{n} (-1)^{i-1} \sum_{1 \leq p_1 < p_2 < \ldots < p_i \leq n} |A_{p_1} \cap A_{p_2} \cap \ldots \cap A_{p_i}|$$

But for any sequence $p = (p_1, \ldots, p_i)$ the intersection $A_{p_1} \cap A_{p_2} \cap \ldots \cap A_{p_i}$ represents all the permutations for which $f(p_j) = j$, for $j \in [i]$. Thus, $|A_{p_1} \cap A_{p_2} \cap \ldots \cap A_{p_i}| = (n-i)!$.

There are $\binom{n}{i}$ possibilities for choosing the sequence p, so finally:

$$|D_n| = n! - \sum_{i=1}^{n} (-1)^{i-1} \binom{n}{i} (n-i)! = \sum_{i=0}^{n} (-1)^i \frac{n!}{i!} = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!}$$

Ratio of Derangements in Permutations

Since $!n = |D_n| = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!}$

The ratio of derangements: $|D_n|/|S_n|$ while $n \to \infty$ tends to $\frac{1}{e}$

$$e^{-1} = \sum_{i=0}^{\infty} (-1)^i \frac{1}{i!} \approx 0.368\ldots,$$

($e \approx 2.7182\ldots$ is the base of the natural logarithm)

Generating Functions

A *generating function* of an infinite sequence $a_0, a_1, \ldots$ is a power series:

$$A(z) = \sum_{i=0}^{\infty} a_i z^i$$

where z is a complex variable

Generating functions is a powerful tool for representing, manipulating and finding closed-form formulas for sequences (especially recurrent sequences)

How to extract a sequence from its generating function?

Let's view $A(z)$ as a function of z, that is convergent in some neighbourhood of z. Then we have:

$$a_k = \frac{A^{(k)}(0)}{k!}$$

(k-th factor in the Maclaurin series of A(z))

Examples

$a_i = [i = n] \rightsquigarrow \sum_{i=0}^{\infty} [i = n] z^i = z^n$

$a_i = c^i \rightsquigarrow \sum_{i=0}^{\infty} c^i z^i = (1 - cz)^{-1}$ (geometric series)

$a_i = [m|i] \rightsquigarrow \sum_{i=0}^{\infty} z^{m \cdot i} = \frac{1}{1-z^m}$

$a_i = (i!)^{-1} \rightsquigarrow \sum_{i=0}^{\infty} \frac{z^i}{i!} = e^z$

$(a) = (0, 1, 1/2, 1/3/, 1/4, \ldots) \rightsquigarrow \sum_{i=1}^{\infty} \frac{z^i}{i} = -ln(1 - z)$

Basic Operations

Let A(z) and B(z) are the generating functions (GF) of sequences (a_i) and (b_i), respectively, $\alpha \in \mathcal{R}$. Then:

- GF of $(a_i + b_i)$ is $A(z) + B(z) = \sum_{i=0}^{\infty}(a_i + b_i)z^i$
- GF of $(\alpha \cdot a_i)$ is $\alpha \cdot A(z) = \sum_{i=0}^{\infty} \alpha \cdot a_i \cdot z^i$
- GF of (a_{i-m}) is $z^m \cdot A(z)$

Further Examples

$$(0,1,0,1\ldots) \rightsquigarrow z(1-z^2)^{-1}$$

$$(1,1/2,1/3,\ldots) \rightsquigarrow -z^{-1}ln(1-z)$$

$$a_i = 1 \rightsquigarrow (1-z)^{-1}$$

$$a_i = (-1)^i \rightsquigarrow (1+z)^{-1}$$

$$i \cdot a_i \rightsquigarrow z \cdot A'(z)$$

$$a_i = i \rightsquigarrow z\frac{d}{dz}(1-z)^{-1} = z(1-z)^{-2}$$

Convolution of sequences

A convolution of two sequences (a_i) and (b_i) is a sequence c_i:

$$c_i = \sum_{k=0}^{i} a_k \cdot b_{i-k}$$

and is denoted as: $(c_i) = (a_i) * (b_i)$
Convolution is commutative.
Fact:

$$\sum_{i=0}^{\infty} c_i z^i = A(z) \cdot B(z)$$

(GF of $(a_i) * (b_i)$ is $A(z) \cdot B(z)$)

Example

Harmonic number:

$$H_n = \sum_{i=1}^{n} \frac{1}{i}$$

Closed-form formula?

GF for (H_n) is a convolution of $(0, 1, 1/2, 1/3, \ldots)$ and $(1, 1, 1, \ldots)$. Thus, this GF is $-(1-z)^{-1} ln(1-z)$.

Example: Closed-form formula Fibonacci Numbers

$$F_i = F_{i-1} + F_{i-2} + [i = 1]$$

thus (its GF is):

$$F(z) = zF(z) + z^2F(z) + z$$

$$F(z) = \frac{z}{1 - z - z^2}$$

$(1 - z - z^2) = (1 - az)(1 - bz)$, where $a = (1 - \sqrt{5})/2$ and $b = (1 + \sqrt{5})/2$. Thus,
$F(z) = \frac{z}{(1-az)(1-bz)} = \frac{1}{(a-b)}(\frac{1}{(1-az)} - \frac{1}{(1-bz)}) = \sum_{i=0}^{\infty} \frac{a^i - b^i}{a-b} \cdot z^i$.
Finally:

$$F_i = \frac{1}{\sqrt{5}}[(\frac{1+\sqrt{5}}{2})^i - (\frac{1-\sqrt{5}}{2})^i]$$

N-th order linear recurrent equations

$$a_i = q(i) + q_1 \cdot a_{i-1} + q_2 \cdot a_{i-2} + \ldots + q_k \cdot a_{i-k}$$

where $q(i) = a_i$, for $i \in [k-1]$ (initial conditions)

$$A(z) = A_0(z) + q_1 \cdot zA(z) + q_2 \cdot z^2 A(z) + \ldots + q_k \cdot z^k A(z)$$

$$A(z) = \frac{a_0 + a_1 z + \ldots + a_{k-1} z^{k-1}}{1 - q_1 z - q_2 z^2 - \ldots - q_k z^k}$$

Thank you for attention